ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 4, pp. 830–849. © Pleiades Publishing, Ltd., 2021.

Construction of 2×2 Fuchsian System with Five Singular Points

S. Rogosin[1*] and L. Khvoshchinskaya[2]

(Submitted by A. M. Elizarov)

[1]*Belarusian State University, Minsk, 220030 Belarus*

[2]*Belarusian State Agrarian Technical University, Minsk, 220023 Belarus*

Received November 14, 2020; revised November 30, 2020; accepted December 14, 2020

Abstract—A constructive method is proposed for the solution of the Riemann–Hilbert problem which is realized in the case of five singular points. The basic components of this method are the logarithmization of the product of matrix functions and the determination of the parameters of a related differential equation of Fuchsian type. The proposed method allows us to solve several related problems, namely, to factorize the piecewise constant matrix functions, to find partial indices and to explicitly solve the vector-matrix boundary value problem with the above piecewise constant matrix function as a coefficient.

DOI: 10.1134/S199508022104017X

Keywords and phrases: *Riemann–Hilbert problem, piecewise constant matrix, differential equation of Fuchs type, monodromy group, logarithmization method, canonical matrix of the boundary value problem, factorization, partial indices.*

Dedicated to the memory
of Leonid Aleksandrovich Aksentiev

1. INTRODUCTION

In 1857, B. Riemann (see, e.g. [29]) posed the following problem: to find (construct) a system of functions $Y(z) = (y_1(z), \ldots, y_m(z))^T$, satisfying three conditions:

1. functions are analytic everywhere in $\widehat{\mathbb{C}}$ except at a finite number of points $a_1, a_2, \ldots, a_n$;
2. the vector $Y(z)$ possesses a linear transformation with a non-singular constant matrix V_k whenever z is orbiting around each singular point a_k $(k = 1, 2, \ldots, n)$, i.e. $Y \rightarrowtail V_k Y$, and $V_1 \cdot V_2 \cdot \ldots \cdot V_n = E$, where E is the $m \times m$ identity matrix;
3. the functions $y_1(z), y_2(z), \ldots, y_m(z)$ possess power type asymptotics of a finite order at each singular point a_k $(k = 1, 2, \ldots, n)$, i.e. $|y_j(z)| \leq C|z - a_k|^{-\alpha}, \forall z, 0 < |z - a_k| < r_k, a_k \neq \infty$ and $|y_j(z)| \leq C|z|^{\alpha}, \forall z, |z| > r_\infty$ for an appropriate $\alpha \geq 0$.

The matrices $V_1, V_2, \ldots, V_n$ generate the so called monodromy group ([2, 5]). Riemann found a representation of the solution in the neighborhood of each particular singular point a_k,

Riemann also pointed out that the functions $y_1, \ldots, y_m$ will be solutions of an m-th order complex differential equation with rational coefficients (see, e.g., [4]). In 1900 Hilbert included the problem of construction of a Fuchsian differential system into his list of mathematical problems for the twentieth century. This question is now known as the twenty-first Hilbert problem or as the Riemann–Hilbert problem, see [5, 8].

*E-mail: rogosin@bsu.by

The Riemann–Hilbert problem can be formulated as the Riemann boundary value problem for analytic vector-functions ([24, 25, 31]). The vector-matrix Riemann boundary value problem with piecewise continuous algebraic coefficients was first solved in [10] by using Green's function method. The solution of the Riemann-Hilbert problem by reduction to the Riemann boundary value problem was proposed in 1908 by Plemelj [27] (see also [28]).

For a long time it was thought that Plemelj had found a complete and positive answer to the question of existence of a complex differential equation with a given monodromy group. Therefore interest in this problem was moved to the effective construction of its solution, the most known are results by Lappo-Danilevsky (functions of matrices method), Röhrl (fibre bundles method) (see, e.g. [5, 3]). Erugin [7] considered the case of four singular points and showed, in particular, that the Riemann–Hilbert problem is related to Painlevé type differential equations (among the recent constructive results we can mention the paper [3]).

The gaps in Plemelj's 1908 paper were first discovered by Kohn [21] and by Arnol'd and Ill'yashenko [2]. In the late 1980s Bolibrukh (see, e.g., [4]) found the first negative answer to the question. In fact, Plemelj's conclusion is valid for the so-called "regular" (see, [4, p. 7]) variant of the Riemann–Hilbert problem. An extended description of the modern state of the Riemann-Hilbert problem as well as the main results by Bolibruch are presented in [4] and [5].

Although this problem has in general a negative solution, there are few cases when an existence of the positive solutions can be proved (for 2×2 Fuchsian systems and any number of singular points this is in Dekkers' theorem [6]). Constructive procedures for solution to the Riemann–Hilbert problem is known only for few cases. Thus, the solution for the systems of two unknown functions ($m = 2$) with three singular points ($n = 3$) is given in [22]). We should also mention the paper [8], cf. [23, 30], which is devoted to the connection between the factorization of piecewise constant $m \times m$ matrix functions with n jumps and the Riemann–Hilbert problem (cases $m = 2$, $n = 4$ and $m = 3$, $n = 3$ are treated there). Several constructive results related to the method developed in this paper are presented in [12–19, 20]. The problem of construction of the differential equation of Fuchsian type is highly related to the determination of accessor parameters which is not considered here, see [1].

The aim of our paper is to determine a procedure which makes the general (small number of unknown functions and arbitrary number of singular points) case constructively solvable. We propose a novel approach (logarithmization method) to the solution of the Riemann–Hilbert problem by bypassing of the commutativity assumption of the generator (see [4]). It generalizes Lappo-Danilevsky's approach and is based on an exact representation for $\ln(V_1 \cdots V_k)$. We limit our attention to the first nontrivial case for which the problem is not solved, namely, for vector functions $Y(z) = (y_1, y_2)^T$ ($m = 2$), and five singular points ($n = 5$). The procedure has been checked to ensure that it is universal, i.e. that it works after slight modifications in the case of six or more singular points and larger m, but its algoritmisation is not yet done.

Within the paper we solve the following problems provided that the monodromy matrices are given. Each of these problems has a particular interest.

1. the logarithmization method is developed, i.e. the method for an exact representation of the logarithm of matrices Sec. 3;
2. explicitly determined, by applying the logarithmization method, so called the differential matrix ([5]) of the system of differential equation, Sec. 4;
3. the above system is reduced to a Fuchsian differential equation of the second order and the fundamental system of its solutions is constructed Subsec. 4.1;
4. the local solution to the Riemann-Hilbert problem in a vicinity of each singular point is determined Subsec. 4.2;
5. the Riemann-Hilbert problem is solved globally by gluing the local solutions into a unique global solution Sec. 5;
6. the partial indices of the piecewise constant matrix are found and a solution to the corresponding homogeneous Riemann boundary value problem with the piecewise constant matrix coefficient and five singular points is constructed Sec. 6.

Solutions of these problems are realized in the case ($m = 2$, $n = 5$). Moreover, intermediate results of Subsec. 3.1 allow us to apply the same technique in the cases $m = 2$, $n = 3$, and $m = 2, n = 4$. The obtained solutions ([12–14]) coincide with those known in the literature. We believe that the general situation can be treated by using our method, it is a goal for our further study.

2. RIEMANN–HILBERT PROBLEM

2.1. Riemann–Hilbert Problem for Two Unknown Functions and Five Singular Points ($m = 2$, $n = 5$)

The Riemann–Hilbert problem in the case $m = 2, n = 5$ consists in the determination of a vector function (or, equivalently, a system of two functions) $Y(z) = (y_1(z), y_2(z))^T$, which is analytic in the extended complex plane $\hat{\mathbb{C}}$ except at five (different) singular points a_1, a_2, a_3, a_4, a_5. This problem we consider in the following class (see, e.g. [24], cf. [6]): $Y(z)$ is supposed to be integrable in neighbourhoods of $a_1, \ldots, a_4$ (more precisely, $|y_j(z)| \leq C|z - a_k|^{-\alpha}$, $0 < |z - a_k| < r_k$) and almost bounded in a neighbourhood of $a_5 = \infty$ (the latter means that $Y(z)$ is either bounded or has a logarithmic singularity at $a_5 = \infty$). Note that the problem in other classes may be unsolvable (cf. [24]).

Let α_k, β_k be eigenvalues of the matrices $V_k, k = 1, \ldots, 5$. We choose the branch of the logarithmic function in such a way that $\rho_k = \frac{1}{2\pi i} \ln \alpha_k, \sigma_k = \frac{1}{2\pi i} \ln \beta_k$ satisfy the conditions (which is possible since Δ is a real number, see e.g. [4])

$$-1 < \mathrm{Re}\, \rho_k \leq 0, \quad -1 < \mathrm{Re}\, \sigma_k \leq 0, \quad \Delta = \sum_{k=1}^{5} (\rho_k + \sigma_k), \quad -9 \leq \Delta \leq 0. \tag{1}$$

Thus, the behaviour of the components of the solution at $a_5 = \infty$ is determined by the numbers $\rho := \rho_5 + k_1$, $\sigma := \sigma_5 + k_2$, where integer numbers k_1, k_2 are chosen in such a way that the so called *Fuchs relation* is satisfied,

$$\sum_{k=1}^{4} (\rho_k + \sigma_k) + \rho + \sigma = 1, \tag{2}$$

which is equivalent to the relation $k_1 + k_2 = 1 - \Delta$. These numbers can be chosen as

$$k_1 = \left[\frac{2-\Delta}{2}\right], \quad k_2 = \left[\frac{1-\Delta}{2}\right] \quad \text{if} \quad \mathrm{Re}\, \rho_5 \leq \mathrm{Re}\, \sigma_5;$$

$$k_1 = \left[\frac{1-\Delta}{2}\right], \quad k_2 = \left[\frac{2-\Delta}{2}\right] \quad \text{if} \quad \mathrm{Re}\, \sigma_5 \leq \mathrm{Re}\, \rho_5. \tag{3}$$

Since the symmetry of relation (2) we can consider without loss of generality that $\mathrm{Re}\, \rho \geq \mathrm{Re}\, \sigma$.

It is known (see, e.g. [4]) that the solution to the Riemann–Hilbert problem can be represented in the locality of each singular point by the following form

$$Y(z) = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = D_k \begin{pmatrix} (z-a_k)^{\rho_k} & u_k(z) \\ (z-a_k)^{\sigma_k} & v_k(z) \end{pmatrix}, \quad 0 < |z - a_k| < r_k, \quad r_k > 0, \quad k = 1, 2, 3, 4, \tag{4}$$

$$Y(z) = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} = D_5 \begin{pmatrix} z^{-\rho} & u_5(z) \\ z^{-\sigma} & v_5(z) \end{pmatrix}, \quad |z| > r_\infty, \quad r_\infty > 0, \tag{5}$$

where D_k $(k = 1, 2, 3, 4, 5)$ transform the monodromy matrices V_k into a normal Jordan form, the functions $u_k(z)$ are analytic in the neighborhoods of points a_k, and the functions $v_k(z)$ are either analytic if $\rho_k \neq \sigma_k$, or have the form

$$v_k(z) = \frac{1}{2\pi i} \ln(z - a_k)\, u_k(z) + w_k(z) \quad \text{if} \quad \rho_k = \sigma_k, \quad k = 1, 2, 3, 4, \tag{6}$$

$$v_5(z) = \frac{1}{2\pi i} \ln z \cdot u_5(z) + w_5(z) \quad \text{if} \quad \rho_5 = \sigma_5, \tag{7}$$

with $w_k(z)$ being analytic in a neighborhood of the point a_k, $u_k(a_k) \neq 0$, $w_k(a_k) \neq 0$, $k = 1, 2, 3, 4, 5$.

2.2. Relation to the Matrix Differential Equation

Let $Y_1(z) = \begin{pmatrix} y_{11}(z) \\ y_{21}(z) \end{pmatrix}$, $Y_2(z) = \begin{pmatrix} y_{12}(z) \\ y_{22}(z) \end{pmatrix}$ be two linear independent solutions to the Riemann–Hilbert problem. Then the matrix $Y(z) = \begin{pmatrix} y_{11}(z) & y_{12}(z) \\ y_{21}(z) & y_{22}(z) \end{pmatrix}$ satisfies the following matrix differential equation with five singular points $a_1, a_2, a_3, a_4, a_5 = \infty$

$$\frac{dY}{dz} = Y \sum_{k=1}^{4} \frac{U_k}{z - a_k}, \tag{8}$$

where U_k are so called *differential matrices* (note that in general the differential equation obtained from the Riemann–Hilbert problem is not Fuchsian, i.e. not necessarily its differential matrix can have not only first order singularities, see e.g. [6]). The constant matrices $U_k, k = 1, 2, 3, 4$ are similar to matrices $W_k = \frac{1}{2\pi i} \ln V_k$ and matrix $U_5 = -\sum_{k=1}^{4} U_k$ is similar to the matrix $S = \begin{pmatrix} -\rho & 0 \\ 0 & 1-\sigma \end{pmatrix}$. Multiplying both sides of equation (8) by a nonsingular constant matrix C we arrive at the following differential equation

$$\frac{d\,(YC)}{dz} = (YC) \sum_{k=1}^{4} \frac{C^{-1} U_k C}{z - a_k}. \tag{9}$$

At the bypasses around the singular points $a_1, \ldots, a_5$ the matrix (YC) is transformed as follows $(YC) \mapsto (V_k Y C)$. Hence the columns of the matrix (YC) are linear independent solutions of the same Riemann–Hilbert problem and this matrix satisfies the differential equation (9) with differential matrices $C^{-1} U_k C$.

Therefore the differential matrices of the Riemann–Hilbert problem can differ only as in similarity transformation. A method of construction of the differential matrices in the case of the Riemann–Hilbert problem with two unknown functions is presented below in Sec. 4.

3. LOGARITHMIZATION METHOD

In this section, we describe in detail the logarithmization method for the product of matrices which allows us to get solution to the Riemann–Hilbert problem not assuming any commutativity of the given matrices. In order to show the essence of the method, we start with the case of the product of two and three matrices (Subsec. 3.1). Further development of the method is presented in the next subsection (Subsec. 3.2). For readers convenience we include the detailed description of the logarithmization method into Appendix at the end of the paper.

3.1. Logarithmization Method for the Product of two and Three Invertible Matrices

Let V_1, V_2 be a constant nonsingular square 2×2 matrices. The equality $\ln(V_1 \cdot V_2) = \ln V_1 + \ln V_2$ is satisfied only if V_1, V_2 are permutation matrices. Now we establish a relationship between the logarithms of matrices $V_1, V_2, V_3 = V_1 \cdot V_2$ for any (non-permutation) matrices.

Proposition 1. *Let α_k, β_k be the eigenvalues of the matrices $V_k, k = 1, 2, 3$ and $\rho_k = \frac{1}{2\pi i} \ln \alpha_k$, $\sigma_k = \frac{1}{2\pi i} \ln \beta_k$ be the eigenvalues of the matrices $W_k = \frac{1}{2\pi i} \ln V_k, k = 1, 2, 3$. Let the branches of the logarithmic functions be fixed in such a way that $|Re\,(\rho_k - \sigma_k)| < 1$, $k = 1, 2$, and the branches for ρ_3, σ_3 be satisfied the following relation*

$$\rho_1 + \sigma_1 + \rho_2 + \sigma_2 = \rho_3 + \sigma_3. \tag{10}$$

Let $\rho_3 \neq \sigma_3$.

Then the matrix $S_3 = \begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$ *can be represented as a sum of two matrices*

$$S_3 = S_1 + S_2, \tag{11}$$

where

$$S_1 = \begin{pmatrix} \frac{\rho_1\sigma_1-(\rho_3-\rho_2)(\rho_3-\sigma_2)}{\sigma_3-\rho_3} & \frac{[(\rho_3-\sigma_1)(\sigma_3-\rho_1)-\rho_2\sigma_2]a}{\sigma_3-\rho_3} \\ \frac{\rho_2\sigma_2-(\rho_3-\rho_1)(\sigma_3-\sigma_1)}{[\sigma_3-\rho_3]a} & \frac{(\sigma_3-\rho_2)(\sigma_3-\sigma_2)-\rho_1\sigma_1}{\sigma_3-\rho_3} \end{pmatrix}, \tag{12}$$

$$S_2 = \begin{pmatrix} \frac{\rho_2\sigma_2-(\rho_3-\rho_1)(\rho_3-\sigma_1)}{\sigma_3-\rho_3} & \frac{[\rho_2\sigma_2-(\rho_3-\sigma_3)(\sigma_3-\rho_1)]a}{\sigma_3-\rho_3} \\ \frac{(\rho_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{[\sigma_3-\rho_3]a} & \frac{(\sigma_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{\sigma_3-\rho_3} \end{pmatrix}, \tag{13}$$

and a *is an arbitrary constant. Here* $S_k \sim W_k, k = 1, 2$ *(*$\sim$ *stands for similarity relation which, in fact, preserves eigenvalues of the matrix).*

Note that the logarithmization method can be applied to any Jordan form of the matrix. Let us show it at the construction of the differential matrix of the Riemann–Hilbert problem with three singular points. In this case it is suitable to use in of $V_3 = V_1V_2$. In this case the Fuchs relation becomes

$$\rho_1 + \sigma_1 + \rho_2 + \sigma_2 + \rho + \sigma = 1. \tag{14}$$

By taken $\rho_3 = -\rho$ and $\sigma_3 = 1 - \sigma$ we can see that (14) coincides with (10). Thus

$$\begin{pmatrix} -\rho & 0 \\ 0 & 1-\sigma \end{pmatrix} = S_1 + S_2, \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = S_1 + S_2 + \begin{pmatrix} -\rho & 0 \\ 0 & -\sigma \end{pmatrix}. \tag{15}$$

If $V_3^0 = \begin{pmatrix} \alpha_3 & 0 \\ 0 & \beta_3 \end{pmatrix}$ is the Jordan form of the matrix V_3, then the matrices $S_3 = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}$ and $S_3' = \begin{pmatrix} \rho & 0 \\ 0 & \sigma-1 \end{pmatrix}$ are different branches of the multivalued function $\frac{1}{2\pi i}\ln V_3^0$, and the matrix $N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is one of the branches of the multivalued function $\frac{1}{2\pi i}\ln E$, moreover

$$N = S_1 + S_2 + S_3. \tag{16}$$

Let us show that representation (15) remains valid if $\alpha_3 = \beta_3$ and the Jordan form of the matrix V_3 has the form $V_3^0 = \begin{pmatrix} \alpha_3 & 0 \\ 1 & \beta_3 \end{pmatrix}$. Then by Lagrange-Sylvester formula [9] $f(V_3^0) = f(\alpha_3)E + f'(\alpha_3)(V_3^0 - \alpha_3E)$ we find $S_3 = \frac{1}{2\pi i}\ln V_3^0 = \begin{pmatrix} \rho & 0 \\ \delta & \rho \end{pmatrix}$, where $\delta = \frac{1}{2\pi i\alpha_3}$. Thus we have in (16) $\begin{pmatrix} -\rho & 0 \\ -\delta & 1-\rho \end{pmatrix} = S_1 + S_2$. Matrix $T = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$ adduce the matrix in the left hand-side to the diagonal form $\begin{pmatrix} -\rho & 0 \\ 0 & 1-\rho \end{pmatrix} = S_1' + S_2'$, where S_1', S_2' have the form (12), (13), respectively, with $\rho_3 = -\rho$, $\sigma_3 = 1-\rho$. Therefore in what follows we can use logarithmization method for any Jordan form of 2×2 matrices.

Corollary 1. *If* $\rho_3 = \rho_1 + \rho_2$, $\sigma_3 = \sigma_1 + \sigma_2$, *i.e the matrices* S_1 *and* S_2 *can be reduced to a triangular form by a unique similarity transformation, then the matrix* S *may be presented in the simple form*

$$S_3 = \begin{pmatrix} \rho_1 & c \\ 0 & \sigma_1 \end{pmatrix} + \begin{pmatrix} \rho_2 & -c \\ 0 & \sigma_2 \end{pmatrix} \quad \text{or} \quad S_3 = \begin{pmatrix} \rho_1 & 0 \\ c & \sigma_1 \end{pmatrix} + \begin{pmatrix} \rho_2 & 0 \\ -c & \sigma_2 \end{pmatrix}.$$

Proposition 2. *Let α_k, β_k be the eigenvalues of the matrices $V_k, k = 1, 2, 3, V_4 = V_1V_2V_3$ and $\rho_k = \frac{1}{2\pi i}\ln\alpha_k, \sigma_k = \frac{1}{2\pi i}\ln\beta_k$ be the eigenvalues of the matrices $W_k = \frac{1}{2\pi i}\ln V_k, k = 1, 2, 3, 4$. Let the branches of the logarithmic functions be fixed as before. Let $\rho_4 \neq \sigma_4$. Then the Jordan form* $S_4 = \begin{pmatrix}\rho_4 & 0\\ 0 & \sigma_4\end{pmatrix}$ of the logarithm of the product of three nonsingular 2×2 matrices can be represented by a sum of logarithms of three matrices which are similar to $W_k, k = 1, 2, 3$, in one of the following form:

$$\begin{pmatrix} s_1 & c\\ \frac{\gamma_1}{c} & \rho_1+\sigma_1-s_1\end{pmatrix} + \begin{pmatrix} s_2 & -(1+\tau)\,c\\ \frac{-\gamma_2}{(1+\tau)c} & \rho_2+\sigma_2-s_2\end{pmatrix} + \begin{pmatrix} s_3 & \tau c\\ \frac{\gamma_3}{\tau c} & \rho_3+\sigma_3-s_3\end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} s_1 & \frac{\gamma_1}{d}\\ d & \rho_1+\sigma_1-s_1\end{pmatrix} + \begin{pmatrix} s_2 & \frac{-\gamma_2}{(1+\tau)d}\\ -(1+\tau)\,d & \rho_2+\sigma_2-s_2\end{pmatrix} + \begin{pmatrix} s_3 & \frac{\gamma_3}{\tau d}\\ \tau d & \rho_3+\sigma_3-s_3\end{pmatrix}, \tag{18}$$

$s_k = \frac{1}{\sigma_4-\rho_4}\left[\rho_4(\sigma_4-\rho_k-\sigma_k)+\rho_k\sigma_k-\tau_k\right], \gamma_k = -(s_k-\rho_k)(s_k-\sigma_k), \tau_k$ @are defined by the formulas $\tau_3 = \rho_{12}\sigma_{12}, \tau_1 = \rho_{23}\sigma_{23}, \tau_2 = \rho_{13}\sigma_{13} = \sum_{k=1}^{4}\rho_k\sigma_k - \tau_1 - \tau_3$, τ is a solution to algebraic equation equation $\gamma_1\tau^2 + (\gamma_1+\gamma_3-\gamma_2)\,\tau + \gamma_3 = 0$, c and d are arbitrary constants.

Remark 1. *Representations (17) and (18), are, in general, equivalent (are equal up to a similarity transformation by a diagonal matrix). If the matrix D_4 transforms V_4 to its Jordan form, and any of the matrices V_k $(k = 1, 2, 3)$ are transformed to their triangular forms, then $\gamma_k = 0$ and we can choose the representation that corresponds to the form of the triangular matrix (upper or lower triangular form).*

Note that as in Proposition 1 we can show that the logarithmization method can be applied to any Jordan form of the matrix W_4.

Corollary 2. *If $\gamma_1 = -(s_1-\rho_1)(s_1-\sigma_1) = 0$, then $c_1d_1 = 0$. The following simpler matrix representations are then possible for S_4:*

$$S_4 = \begin{pmatrix}\rho_1 & 0\\ \frac{\gamma_2-\gamma_3}{c} & \sigma_1\end{pmatrix} + \begin{pmatrix} s_2 & -c\\ \frac{-\gamma_2}{c} & \rho_2+\sigma_2-s_2\end{pmatrix} + \begin{pmatrix} s_3 & c\\ \frac{\gamma_3}{c} & \rho_3+\sigma_3-s_3\end{pmatrix},$$

$$S_4 = \begin{pmatrix}\rho_1 & \frac{\gamma_2-\gamma_3}{d}\\ 0 & \sigma_1\end{pmatrix} + \begin{pmatrix} s_2 & \frac{-\gamma_2}{d}\\ -d & \rho_2+\sigma_2-s_2\end{pmatrix} + \begin{pmatrix} s_3 & \frac{\gamma_3}{d}\\ d & \rho_3+\sigma_3-s_3\end{pmatrix},$$

$$S_4 = \begin{pmatrix}\rho_1 & 0\\ 0 & \sigma_1\end{pmatrix} + \begin{pmatrix} s_2 & \frac{-\gamma_3}{d}\\ -d & \rho_2+\sigma_2-s_2\end{pmatrix} + \begin{pmatrix} s_3 & \frac{\gamma_3}{d}\\ d & \rho_3+\sigma_3-s_3\end{pmatrix}.$$

@Similar representations occur in the cases $\gamma_2 = 0, \gamma_3 = 0$.

3.2. Logarithmization Method for the Product of Four Invertible Matrices

Let us present the matrix $S_5 = \begin{pmatrix}-\rho & 0\\ 0 & 1-\sigma\end{pmatrix}$ as a sum of four matrices $S_5 = S_1 + S_2 + S_3 + S_4$, where $S_k \sim W_k = \frac{1}{2\pi i}\ln V_k$, $k = 1, 2, 3, 4$. For this, we rewrite the matrix $V_5 := V_1 \cdot V_2 \cdot V_3 \cdot V_4$ as the following products of three matrices

$$V_5 = V_1 \cdot V_2 \cdot (V_3 \cdot V_4) = V_1 \cdot V_2 \cdot V_{34}, \tag{19}$$

$$V_5 = (V_1 \cdot V_2) \cdot V_3 \cdot V_4 = V_{12} \cdot V_3 \cdot V_4, \tag{20}$$

and apply formulas (17), (18). For further analysis we need to know not only the eigenvalues α_k, β_k of the monodromy matrices V_k, but also the eigenvalues of the products V_1V_2, V_2V_3, V_3V_4, $V_1V_2V_3$, $V_2V_3V_4$ of these matrices and their logarithms.

We denote by $\alpha_{k,k+1}$, $\beta_{k,k+1}$ the eigenvalues of $V_{k,k+1} = V_kV_{k+1}$, $k = 1, 2, 3$, and by $\alpha_{k,k+1,k+2}$, $\beta_{k,k+1,k+2}$ the eigenvalues of $V_{k,k+1,k+2} = V_kV_{k+1}V_{k+2}$, $k = 1, 2$. We determine the following parameters $\rho_{k,k+1} = \frac{1}{2\pi i}\ln\alpha_{k,k+1}$, $\sigma_{k,k+1} = \frac{1}{2\pi i}\ln\beta_{k,k+1}$, $\rho_{k,k+1,k+2} = \frac{1}{2\pi i}\ln\alpha_{k,k+1,k+2}$, $\sigma_{k,k+1,k+2} = \frac{1}{2\pi i}\ln\beta_{k,k+1,k+2}$, in such a way that branches of the logarithmic functions are fixed according to the conditions

$$\rho_{k,k+1} + \sigma_{k,k+1} = \rho_k + \rho_{k+1} + \sigma_k + \sigma_{k+1}, \quad |\mathrm{Re}\,(\rho_{k,k+1} - \sigma_{k,k+1})| < 1, \quad k = 1, 2, 3,$$
$$\rho_{k,k+1,k+2} + \sigma_{k,k+1,k+2} = \rho_k + \rho_{k+1} + \rho_{k+2} + \sigma_k + \sigma_{k+1} + \rho_{k+2},$$
$$|\mathrm{Re}\,(\rho_{k,k+1,k+2} - \sigma_{k,k+1,k+2})| < 1, \quad k = 1, 2.$$

Proposition 3. *Let α_k, β_k be the eigenvalues of the matrices V_k, $k = 1, 2, 3, 4$, and $\rho_k = \frac{1}{2\pi i}\ln\alpha_k$, $\sigma_k = \frac{1}{2\pi i}\ln\beta_k$ be the eigenvalues of the matrices $W_k = \frac{1}{2\pi i}\ln V_k$, $k = 1, 2, 3, 4$. Let the branches of the logarithmic functions are fixed as above. Let $\rho_5 \neq \sigma_5$.*

Then the Jordan form $S_5 = \begin{pmatrix}\rho_5 & 0\\ 0 & \sigma_5\end{pmatrix}$ of the logarithm of the product of four nonsingular matrices of the second order can be represented by a sum of logarithms of three matrices which are similar to W_k, $k = 1, 2, 3, 4$:

$$S_5 = \begin{pmatrix} s_1 & c \\ \frac{\gamma_1}{c} & \rho_1 + \sigma_1 - s_1 \end{pmatrix} + \begin{pmatrix} s_2 & -(1+\tau_1)c \\ -\frac{\gamma_2}{(1+\tau_1)c} & \rho_3 + \sigma_3 - s_3 \end{pmatrix}$$
$$+ \begin{pmatrix} s_3 & \tau_1(1+\tau_2)c \\ \frac{\gamma_3}{\tau_1(1+\tau_2)c} & \rho_3 + \sigma_3 - s_3 \end{pmatrix} + \begin{pmatrix} s_4 & -\tau_1\tau_2 c \\ -\frac{\gamma_4}{\tau_1\tau_2 c} & \rho_4 + \sigma_4 - s_4 \end{pmatrix}, \tag{21}$$

where τ_1, τ_2 are solutions to $\gamma_1\tau_1^2 + (\gamma_1 + \gamma_{34} - \gamma_2)\tau_1 + \gamma_{34} = 0$, $\gamma_{12}\tau_2^2 + (\gamma_{12} + \gamma_4 - \gamma_3)\tau_2 + \gamma_4 = 0$, respectively, and c is an arbitrary constant.

If we replace τ_1 by $\frac{1}{\tau_1}$ and take into account that $\gamma_{12} = \gamma_{34}$, then we arrive at another presentation of the matrix S_5:

$$S_5 = \begin{pmatrix} s_1 & \tau_1 c \\ \frac{\gamma_1}{\tau_1 c} & \rho_1 + \sigma_1 - s_1 \end{pmatrix} + \begin{pmatrix} s_2 & -(1+\tau_1)c \\ -\frac{\gamma_2}{(1+\tau_1)c} & \rho_3 + \sigma_3 - s_3 \end{pmatrix}$$
$$+ \begin{pmatrix} s_3 & (1+\tau_2)c \\ \frac{\gamma_3}{(1+\tau_2)c} & \rho_3 + \sigma_3 - s_3 \end{pmatrix} + \begin{pmatrix} s_4 & -\tau_2 c \\ -\frac{\gamma_4}{\tau_2 c} & \rho_4 + \sigma_4 - s_4 \end{pmatrix}$$
$$= S_1' + S_2' + S_3' + S_4', \tag{22}$$

as well as at the equivalent presentation

$$S_5 = \begin{pmatrix} s_1 & \frac{\gamma_1}{\tau_1 d} \\ \tau_1 d & \rho_1 + \sigma_1 - s_1 \end{pmatrix} + \begin{pmatrix} s_2 & -\frac{\gamma_2}{(1+\tau_1)d} \\ -(1+\tau_1)d & \rho_3 + \sigma_3 - s_3 \end{pmatrix}$$
$$+ \begin{pmatrix} s_3 & \frac{\gamma_3}{(1+\tau_2)d} \\ (1+\tau_2)d & \rho_3 + \sigma_3 - s_3 \end{pmatrix} + \begin{pmatrix} s_4 & -\frac{\gamma_4}{\tau_2 d} \\ -\tau_2 d & \rho_4 + \sigma_4 - s_4 \end{pmatrix}$$
$$= S_1'' + S_2'' + S_3'' + S_4'', \tag{23}$$

where c and d are arbitrary constants,

$$s_k = \frac{\rho(\sigma + \rho_k + \sigma_k - 1) + \omega_k - \omega_k^*}{1 + \rho - \sigma}, \quad k = 1, 2, 3, 4,$$
$$\omega_1^* = \omega_{234}, \quad \omega_2^* = \omega_{134}, \quad \omega_3^* = \omega_{124}, \quad \omega_4^* = \omega_{123},$$

and τ_1, τ_2 are solutions of the following equations, respectively,

$$\gamma_{12}\tau_1^2 + (\gamma_{12} + \gamma_1 - \gamma_2)\tau_1 + \gamma_1 = 0, \quad \gamma_{12}\tau_2^2 + (\gamma_{12} + \gamma_4 - \gamma_3)\tau_2 + \gamma_4 = 0. \tag{24}$$

It follows from (22) that the above presentation of the matrix S is determined uniquely up to similarity transformation via a diagonal matrix. Therefore, the matrices in (22) and (23) are differential matrices of the equation (8). It can be shown that the statement remains valid for another Jordan form of the matrix W_5.

4. FUCHSIAN DIFFERENTIAL EQUATION WITH FIVE SINGULAR POINTS

4.1. Construction of the Differential Equation

Let us now construct the second order differential equation of the Fuchsian class, related to the following matrix differential equation

$$\frac{dY}{dz}(z) = Y(z)\sum_{k=1}^{4}\frac{S_k}{z - a_k}, \quad \text{or} \quad \frac{dY}{dz} = Y(z)S(z), \tag{25}$$

with respect to the unknown matrix $Y(z) = (y_{ij}(z))$ and matrices S_k as presented in (21), where $S(z) = (s_{ij}(z))$. Thus

$$s_{11}(z) = \sum_{k=1}^{4}\frac{s_k}{z - a_k}, s_{12}(z) = \sum_{k=1}^{4}\frac{c_k}{z - a_k}, s_{21}(z) = \sum_{k=1}^{4}\frac{d_k}{z - a_k}, s_{22}(z) = \sum_{k=1}^{4}\frac{\rho_k - \sigma_k - s_k}{z - a_k},$$
$$\sum_{k=1}^{4} c_k = 0, \sum_{k=1}^{4} d_k = 0, s_k' = \rho_k - \sigma_k - s_k, s_k s_k' - c_k d_k = \rho_k\sigma_k.$$

Matrix equation (25) can be written in the following scalar form

$$\begin{cases} y_{11}' = s_{11}y_{11} + s_{21}y_{12}, \\ y_{12}' = s_{12}y_{11} + s_{22}y_{12}, \\ y_{21}' = s_{11}y_{21} + s_{21}y_{22}, \\ y_{22}' = s_{12}y_{21} + s_{22}y_{22}. \end{cases} \tag{26}$$

By changing the first equation of (26) to express y_{12},

$$y_{12} = \frac{1}{s_{21}}\left(y_{11}' - s_{11}y_{11}\right) \tag{27}$$

and substituting it into the second equation, we arrive at the following differential equation:

$$y'' - \left(s_{11} + s_{22} + \frac{s_{21}'}{s_{21}}\right)y' + \left(s_{11}s_{22} - s_{12}s_{21} - s_{11}' + \frac{s_{21}'}{s_{21}}s_{11}\right)y = 0. \tag{28}$$

It follows from the third and the fourth equations of (26) that the function y_{21} is also a solution to equation (28).

Now we reformulate equation (28), using the forms of the matrices S_k in (21). Note that $\frac{s_{21}'}{s_{21}} = (\ln s_{21})' = \left(\ln \sum\limits_{k=1}^{4}\frac{d_k}{z-a_k}\right)'$. By direct calculation we obtain

$$\frac{s_{21}'}{s_{21}} = \frac{1}{z - b_1} + \frac{1}{z - b_2} + \sum_{k=1}^{4}\frac{1}{z - a_k}.$$

where b_1, b_2 are the roots of the following quadric equation

$$\left(\sum_{k=1}^{4} a_k d_k\right) z^2 - \left(\sum_{k=1}^{4} a_k d_k (a_1 + a_2 + a_3 + a_4 - a_k)\right) z - a_1 a_2 a_3 a_4 \left(\sum_{k=1}^{4} \frac{d_k}{a_k}\right) = 0. \tag{29}$$

Since $d_1 = d\tau_1$, $d_2 = -d(1+\tau_1)$, $d_3 = d(1+\tau_2)$, $d_4 = -d\tau_2$ we can rewrite (29) in the form

$$(a_3 - a_2 + \tau_1(a_1 - a_2) + \tau_2(a_3 - a_4))\, z^2 - ((a_3 - a_2)(a_1 + a_4) + \tau_1(a_1 - a_2)(a_3 + a_4$$
$$+\tau_2(a_3 - a_4)(a_1 + a_2))\, z + ((a_3 - a_2)a_1 a_4 + \tau_1(a_1 - a_2)a_3 a_4 + \tau_2(a_3 - a_4)a_1 a_2) = 0.$$

Further, we have

$$s_{11} + s_{22} + \frac{s_{21}'}{s_{21}} = \sum_{k=1}^{4} \frac{\rho_k + \sigma_k - 1}{z - a_k} + \frac{1}{z - b_1} + \frac{1}{z - b_2},$$

$$s_{11}s_{22} - s_{12}s_{21} = \frac{s_1 s_1' - c_1 d_1}{(z-a_1)^2} + \frac{s_1 s_2' + s_1' s_2 - c_1 d_2 - c_2 d_1}{(z-a_1)(z-a_2)} + \cdots$$

$$= \frac{\rho_1\sigma_1}{(z-a_1)^2} + \frac{(s_1+s_2)(s_1'+s_2') - (c_1+c_2)(d_1+d_2) - s_1 s_1' - s_2 s_2' + c_1 d_1 + c_2 d_2}{(z-a_1)(z-a_2)} + \cdots$$

$$= \frac{\rho_1\sigma_1}{(z-a_1)^2} + \frac{\omega_{12} - \rho_1\sigma_1 - \rho_2\sigma_2}{(z-a_1)(z-a_2)} + \cdots = \sum_{k=1}^{4} \frac{\omega_k}{(z-a_k)^2} + \sum_{k,j=1,k\neq j}^{4} \frac{\omega_{kj} - s_{kj} - \omega_k - \omega_j}{(z-a_k)(z-a_j)},$$

$$-s_{11}' + \frac{s_{21}'}{s_{21}} s_{11} = \left(\frac{1}{z-b_1} + \frac{1}{z-b_2}\right) \sum_{k=1}^{4} \frac{s_k}{z-a_k} - \sum_{k,j=1,k\neq j}^{4} \frac{s_k + s_j}{(z-a_k)(z-a_j)}.$$

It follows from Proposition 3 that

$$s_1 + s_2 = \frac{[\rho(\sigma + \rho_{12} + \sigma_{12} - 1) + \omega_{12} - \omega_{34}]}{1 + \rho - \sigma} = s_{12}.$$

Analogously

$$s_k + s_j = \frac{[\rho(\sigma + \rho_{kj} + \sigma_{kj} - 1) + \omega_{kj} - \omega_{kj}^{**}]}{1 + \rho - \sigma} = s_{kj}, \tag{30}$$

where

$$\omega_{12}^{**} = \omega_{34}, \quad \omega_{34}^{**} = \omega_{12}, \quad \omega_{14}^{**} = \omega_{23}, \quad \omega_{13}^{**} = \omega_{24}, \quad \omega_{23}^{**} = \omega_{14}, \quad \omega_{24}^{**} = \omega_{13}. \tag{31}$$

Substituting the obtained formulas into (28), we arrive at the following form for the solution of the Riemann–Hilbert problem with five singular points

Theorem 1. *Second order differential equation of the Fuchsian type, with five singular points and a given monodromy group, can be presented in the form*

$$y''(z) + \left(\sum_{k=1}^{4} \frac{1 - \rho_k - \sigma_k}{z - a_k} - \frac{1}{z-b_1} - \frac{1}{z-b_2}\right) y'(z) + \left(\sum_{k=1}^{4} \frac{\omega_k}{(z-a_k)^2}\right.$$
$$\left. + \sum_{k,j=1,k\neq j}^{4} \frac{\omega_{kj} - s_{kj} - \omega_k - \omega_j}{(z-a_k)(z-a_j)} + \left(\frac{1}{z-b_1} + \frac{1}{z-b_2}\right) \sum_{k=1}^{4} \frac{s_k}{z-a_k}\right) y(z) = 0. \tag{32}$$

4.2. Local Solution to the Differential Equation

In a neighbourhood of each of the singular points $a_1, \ldots, a_5$, equation (32) has two independent solutions (see e.g. [11], cf. [7])

$$u_k(z) = (z - a_k)^{\rho_k} \sum_{n=0}^{\infty} B_n^{(k)} (z - a_k)^n,$$

$$v_k(z) = (z-a_k)^{\sigma_k}\sum_{n=0}^{\infty} C_n^{(k)}(z-a_k)^n, \quad \text{whenever} \quad \rho_k \neq \sigma_k,$$

$$v_k(z) = \frac{1}{2\pi i}\ln(z-a_k)u_k(z) + (z-a_k)^{\rho_k} w_k(z), \quad \text{whenever} \quad \rho_k = \sigma_k, k=1,\ldots,4, \tag{33}$$

$$u_5(z) = z^{-\rho}\sum_{n=0}^{\infty} B_n^{(5)} z^{-n},$$

$$v_5(z) = z^{-\sigma}\sum_{n=0}^{\infty} C_n^{(5)} z^{-n}, \quad \text{whenever} \quad \rho \neq \sigma$$

$$v_5(z) = \frac{1}{2\pi i}\ln z u_k(z) + z^{-\rho} w_5(z), \quad \text{whenever} \quad \rho = \sigma. \tag{34}$$

Here the functions $w_k(z)$ are analytic in the vicinity of the singular points $a_k, k = 1,\ldots,5$, and the coefficients $B_n^{(k)}, C_n^{(k)}$ are defined from recurrence relations, after substitution into equation (32). In the vicinity of points b_1 and b_2, equation (32) has two linear independent solutions

$$u_{b_k}(z) = \sum_{n=0}^{\infty} B_n^{(b_k)}(z-b_k)^n, \quad v_{b_k}(z) = (z-b_k)^2\sum_{n=0}^{\infty} C_n^{(b_k)}(z-b_k)^n, \quad k=1,2. \tag{35}$$

The point b_1 and/or b_2 might coincide with one of the singular points $a_k (k = 1,\ldots,4)$. If the matrix S_1 is triangular, then $\gamma_1 = 0$ and $d_1 = 0$. Then if by linear transformation the point a_1 corresponds to 0, then $z = 0$ is a root of equation (29) and $b_1 = a_1$ whenever $\gamma_1 = 0$.

Now we present the elements y_{12}, y_{22} of the matrix $Y(z)$ in terms of the elements y_{11}, y_{21}. Substituting expressions for s_{21}, s_{11} into (32) we then obtain

$$y_{12} = \frac{1}{dr}\frac{\prod_{k=1}^{4}(z-a_k)}{(z-b_1)(z-b_2)}\left(y_{11}' - \sum_{k=1}^{4}\frac{s_k}{z-a_k}y_{11}\right) = \frac{1}{d}\frac{p(z)}{q(z)}\left(y_{11}' - \sum_{k=1}^{4}\frac{s_k}{z-a_k}y_{11}\right),$$

where

$$p(z) = \prod_{k=1}^{4}(z-a_k), \quad q(z) = r(z-b_1)(z-b_2) = (a_3 - a_2 + \tau_1(a_1-a_2) + \tau_2(a_3-q_4))\,z^2$$
$$-\left((a_3-a_2)(a_1+a_4) + \tau_1(a_1-a_2)(a_3+a_4) + \tau_2(a_3-a_4)(a_1+a_2)\right) z$$
$$+\left((a_3-a_2)a_1a_4 + \tau_1(a_1-a_2)a_3a_4 + \tau_2(a_3-a_4)a_1a_2\right).$$

Analogously,

$$y_{22} = \frac{1}{dr}\frac{\prod_{k=1}^{4}(z-a_k)}{(z-b_1)(z-b_2)}\left(y_{21}' - \sum_{k=1}^{4}\frac{s_k}{z-a_k}y_{21}\right) = \frac{1}{d}\frac{p(z)}{q(z)}\left(y_{21}' - \sum_{k=1}^{4}\frac{s_k}{z-a_k}y_{21}\right).$$

Therefore, if u_k, v_k is a fundamental system of solutions to differential equation (32) in a neighbourhood of the point $a_k, k = 1,\ldots,5$, then the solution $Y(z)$ to the matrix equation (25) in this neighbourhood can be written in the form

$$Y(z) = D_k\begin{pmatrix} u_k & \frac{p(z)}{q(z)}\left(u_k' - \sum_{k=1}^{4}\frac{s_k}{z-a_k}u_k\right) \\ v_k & \frac{p(z)}{q(z)}\left(v_k' - \sum_{k=1}^{4}\frac{s_k}{z-a_k}v_k\right)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix} = X(z)\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix}, \tag{36}$$

where D_k are the matrices transforming matrices $V_k, k = 1,\ldots,5$, to their normal Jordan forms. The order h of the determinant of the matrix $X(z)$ at infinity is

$$h = \operatorname{Re}\rho + (\operatorname{Re}\sigma + 1) - 2 = \operatorname{Re}\rho + \operatorname{Re}\sigma - 1,$$

and the order of the first column of the matrix $X(z)$ is $h_1 = \min\{\mathrm{Re}\,\rho, \mathrm{Re}\,\sigma\} = \mathrm{Re}\,\rho$. Since $s_1 + s_2 + s_3 + s_4 = -\rho$, then the order of the second column of $X(z)$ is $h_2 = \min\{\mathrm{Re}\,\rho, \mathrm{Re}\,\sigma - 1\} = \mathrm{Re}\,\sigma - 1$. Hence $h = h_1 + h_2$. Therefore we arrive at the following result (cf. [24]).

Theorem 2. *Let $u_k, v_k, k = 1, \ldots, 5$, be a linear independent solutions of equation (32) in a neighbourhood of the corresponding singular point a_k, represented in one of the forms (33), (34). Let $p(z) = \prod_{k=1}^{4}(z - a_k), q(z) = r(z - b_1)(z - b_2)$ and b_1, b_2 be the roots of the quadric equation (29). Then the matrix*

$$X(z) = D_k \begin{pmatrix} u_k & \frac{p(z)}{q(z)}\left(u_k' - \sum\limits_{k=1}^{4} \frac{s_k}{z-a_k} u_k\right) \\ v_k & \frac{p(z)}{q(z)}\left(v_k' - \sum\limits_{k=1}^{4} \frac{s_k}{z-a_k} v_k\right) \end{pmatrix} \tag{37}$$

is a local solution to the Riemann–Hilbert problem (i.e. solves equation (25)) in a neighbourhood of each singular point $a_k, k = 1, \ldots, 5$. The matrix $X(z)$ meets the following conditions:

a) $\det X(z) \neq 0, \forall z \neq a_k, k = 1, \ldots, 5$;

b) the columns of $X(z)$ belong to a chosen class of functions, specifically they are supposed to be integrable in a neighbourhood of $a_1, \ldots, a_4$ and almost bounded in a neighbourhood of $a_5 = \infty$;

c) the order of $\det X(z)$ *at infinity is equal to the sum of the orders of its columns.*

5. GLOBAL SOLUTION TO THE RIEMANN–HILBERT PROBLEM

The obtained solution exists not only in a neighbourhood of each singular point. An analytic continuation of a local solution in a neighbourhood of a certain singular point does not necessarily become the solution at another singular point. Thus, in order to obtain a solution over the whole complex plane (i.e. the global solution), we need to relate the local representations to each other (i.e. assemble these representations into a unique analytic solution). Note that analytic continuation of the local solution corresponding a_5 coincides with the starting local solution corresponding to a_1 due to fulfillment of the Fuchs relation. Possible direct calculation can be provided too.

The solution (37) to equation (25) in a neighbourood of each singular point a_k is determined up to two constants, since the matrix D_k that transforms V_k to a normal Jordan form is not defined uniquely and has the form

$$D_k = \tilde{D}_k T_k, \quad \text{where} \quad T_k = \begin{cases} \begin{pmatrix} \delta_k & 0 \\ 0 & \varepsilon_k \end{pmatrix}, & \text{if} \quad \alpha_k \neq \beta_k, \\ \begin{pmatrix} \delta_k & 0 \\ \varepsilon_k & \frac{\delta_k}{\alpha_k} \end{pmatrix}, & \text{if} \quad \alpha_k = \beta_k, \end{cases} \tag{38}$$

where $\tilde{D}_k$ are fixed matrices transforming V_k to a normal Jordan forms, and the constants δ_k, ε_k are subjects for further determination. The fundamental system of solutions to differential equation (32) is also determined up to two constant multipliers. Without loss of generality we fix values of the coefficients $B_0^{(k)}, C_0^{(k)}, k = 1, \ldots, 5$, in (33), (34), e.g. $B_0^{(k)} = 1, C_0^{(k)} = 1, k = 1, \ldots, 5$. Since equation (32) has only two linear independent solutions, the solutions u_k, v_k and u_{k+1}, v_{k+1} satisfy the following linear relations:

$$\begin{pmatrix} u_k(z) \\ v_k(z) \end{pmatrix} = \Lambda_k \begin{pmatrix} u_{k+1}(z) \\ v_{k+1}(z) \end{pmatrix}, \tag{39}$$

where $\Lambda_k\,(\lambda_{ij})$ are constant nonsingular matrices whose elements can be found via the formulas

$$\Lambda_k = \begin{pmatrix} u_k(z_0) & u_k'(z_0) \\ v_k(z_0) & v_k'(z_0) \end{pmatrix} \begin{pmatrix} u_{k+1}(z_0) & u_{k+1}'(z_0) \\ v_{k+1}(z_0) & v_{k+1}'(z_0) \end{pmatrix}^{-1},$$

and z_0 is a fixed point in a common part of the convergence domains for the corresponding series. Local solutions in neighbourhoods of points a_k and a_{k+1} are analytic continuation of each other iff $D_k \begin{pmatrix} u_k(z) \\ v_k(z) \end{pmatrix} = D_k\Lambda_k \begin{pmatrix} u_{k+1}(z) \\ v_{k+1}(z) \end{pmatrix} = D_{k+1} \begin{pmatrix} u_{k+1}(z) \\ v_{k+1}(z) \end{pmatrix}$ or, in notation from (38) $\tilde{D}_k T_k = \tilde{D}_{k+1} T_{k+1}$. Denoting $M_k \left(\mu_{ij}^{(k)}\right) = \tilde{D}_{k+1}^{-1}\tilde{D}_k$, we define δ_k, ε_k as being from the system $T_k\Lambda_k = M_k T_{k+1}, k = 1, \ldots, 4$. In particular, if $\rho_1 \neq \sigma_1$ and $\rho_2 \neq \sigma_2$, then such a system has the form

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \varepsilon_1 \end{pmatrix} \begin{pmatrix} \lambda_{11}^{(1)} & \lambda_{12}^{(1)} \\ \lambda_{21}^{(1)} & \lambda_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} \mu_{11}^{(1)} & \mu_{12}^{(1)} \\ \mu_{21}^{(1)} & \mu_{22}^{(1)} \end{pmatrix} \begin{pmatrix} \delta_2 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}. \tag{40}$$

The solvability condition for system (40) has the form

$$\lambda_{11}^{(1)}\lambda_{22}^{(1)}\mu_{12}^{(1)}\mu_{21}^{(1)} = \lambda_{12}^{(1)}\lambda_{21}^{(1)}\mu_{11}^{(1)}\mu_{22}^{(1)}. \tag{41}$$

If matrices V_1 and V_2 are transformed by the same similarity transformation to the triangular form, then relation (41) holds automatically. If not then we may rewrite (41) in the form

$$\frac{\lambda_{11}^{(1)}\lambda_{22}^{(1)}}{\lambda_{12}^{(1)}\lambda_{21}^{(1)}} = \frac{\mu_{11}^{(1)}\mu_{22}^{(1)}}{\mu_{12}^{(1)}\mu_{21}^{(1)}}. \tag{42}$$

Since $\rho_1 \neq \sigma_1$ and $\rho_2 \neq \sigma_2$, we can then take matrices $\tilde{D}_k$ in one of the forms

$$\tilde{D}_k = \begin{pmatrix} \nu_{12}^{(k)} & \nu_{12}^{(k)} \\ \alpha_k - \nu_{11}^{(k)} & \beta - \nu_{11}^{(k)} \end{pmatrix} \quad \text{or} \quad \tilde{D}_k = \begin{pmatrix} \alpha_k - \nu_{22}^{(k)} & \beta - \nu_{22}^{(k)} \\ \nu_{21}^{(k)} & \nu_{21}^{(k)} \end{pmatrix}, \quad k = 1, 2;$$

$$M_1 = \frac{1}{\nu_{12}^{(1)}(\beta_1 - \alpha_1)} \begin{pmatrix} (\beta_1 - \nu_{11}^{(1)})\nu_{12}^{(1)} - (\alpha_2 - \nu_{11}^{(2)})\nu_{12}^{(1)} & (\beta_1 - \nu_{11}^{(1)})\nu_{12}^{(2)} - (\alpha_2 - \nu_{11}^{(2)})\nu_{12}^{(1)} \\ -(\alpha_1 - \nu_{11}^{(1)})\nu_{12}^{(2)} + (\alpha_2 - \nu_{11}^{(2)})\nu_{12}^{(1)} & -(\alpha_1 - \nu_{11}^{(1)})\nu_{12}^{(2)} + (\beta_2 - \nu_{11}^{(2)})\nu_{12}^{(1)} \end{pmatrix}.$$

Hence by simple algebra we get

$$\mu_{11}^{(1)}\mu_{22}^{(1)} = \nu_{12}^{(2)}(\nu_{12}^{(1)})^{-1}(\beta_1 - \alpha_1)^{-2}(\alpha_{12} + \beta_{12} - \alpha_1\beta_2 - \alpha_2\beta_1),$$
$$\mu_{12}^{(1)}\mu_{21}^{(1)} = \nu_{12}^{(2)}(\nu_{12}^{(1)})^{-1}(\beta_1 - \alpha_1)^{-2}(\alpha_{12} + \beta_{12} - \alpha_1\alpha_2 - \beta_1\beta_2).$$

Hence condition (42) can be rewritten as

$$\frac{\lambda_{11}^{(1)}\lambda_{22}^{(1)}}{\lambda_{12}^{(1)}\lambda_{21}^{(1)}} = \frac{\alpha_{12} + \beta_{12} - (\alpha_1\beta_2 + \alpha_2\beta_1)}{\alpha_{12} + \beta_{12} - (\alpha_1\alpha_2 + \beta_1\beta_2)}. \tag{43}$$

We note that neither the numerator nor the denominator of the ratio in (43) is non-vanishing. Moreover, this ratio can not equal unity due to the non-singularity of matrices Λ_1 and M_1. Direct observation shows that the recurrent relations for coefficients $B_n^{(1)}$ and $C_n^{(1)}$ contain in the denominators only ρ_1, σ_1 and do not contain the other pairs $\rho_2, \sigma_2; \rho_3, \sigma_3; \rho_4, \sigma_4; \ldots, \rho, \sigma$. Analogously, recurrent relations for coefficients $B_n^{(2)}$ and $C_n^{(2)}$ contain in the denominators only ρ_2, σ_2 and do not contain the other pairs $\rho_1, \sigma_1; \rho_3, \sigma_3; \rho_4, \sigma_4; \ldots, \rho, \sigma$. Therefore, the function $f(\rho_1, \sigma_1; \rho_2, \sigma_2; \rho_3, \sigma_3; \rho_4, \sigma_4; \rho_{12}, \sigma_{12}; \ldots; \rho_{123}, \sigma_{123}; \ldots; \rho, \sigma) = \frac{\lambda_{11}^{(1)}\lambda_{22}^{(1)}}{\lambda_{12}^{(1)}\lambda_{21}^{(1)}}$ is an entire function with respect to all of its variables excluding $\rho_1, \sigma_1; \rho_2, \sigma_2; \rho_{12}, \sigma_{12}$ and does not include the values $0, 1, \infty$. Hence, the function f depends only on $\rho_1, \sigma_1; \rho_2, \sigma_2; \rho_{12}, \sigma_{12}$.

Substitute the following into (32): $\rho_3 = \sigma_3(\omega_3 = 0); \rho_4 = \sigma_4(\omega_4 = 0);$ $\rho_{34} = \sigma_{34}(\omega_{34} = 0); \rho_{23} = \rho_{234} = \rho_2; \sigma_{23} = \sigma_{234} = \sigma_2(\omega_{23} = \omega_{234} = \omega_2)$. The Fuchs relation (2) then becomes $\rho_{12} + \sigma_{12} + \rho +$

$\sigma = 1$. Hence we have $\rho = -\rho_{12}, \sigma = 1 - \sigma_{12}$. It then follows $\omega_{13} = \omega_{14} = \omega_1$, $\omega_{23} = \omega_2$, $\omega = \omega_{123} = \omega_{13}$. Therefore

$$s_{12} = \frac{[\rho(\sigma + \rho_{12} + \sigma_{12} - 1) + \omega_{12} - \omega_{34}}{1 + \rho - \sigma} = \rho_{12}; \quad s_3 = \frac{[\rho(\sigma + \rho_3 + \sigma_3 - 1) + \omega_3 - \omega_{124}}{1 + \rho - \sigma} = 0;$$

$$s_4 = \frac{[\rho(\sigma + \rho_4 + \sigma_4 - 1) + \omega_4 - \omega_{123}}{1 + \rho - \sigma} = 0;$$

$$\gamma_{12} = -(s_{12} - \rho_{12})(s_{12} - \sigma_{12}) = 0, \quad \gamma_{34} = \gamma_{12} = 0, \quad \gamma_3 = \gamma_4 = 0.$$

From the first equation (24) we have $\tau_1 = \frac{\gamma_2}{\gamma_2 - \gamma_1}$, moreover the second equation is satisfied identically. It then follows from representation (22) that in equation (29) we have $d_1 + d_2 = 0, d_3 = d_4 = 0$, and that equation (29) can be written in the form

$$(a_1 - a_2)z^2 + (a_1 - a_2)(a_3 + a_4)z + (a_1 - a_2)a_3 a_4 = 0,$$

whose roots are $z_1 = a_3, z_2 = a_4$. Thus we have from equation (32) that $b_1 = a_3$, $b_2 = a_4$, $s_{12} = \rho_{12}$, $s_{13} = s_1 + s_3 = s_1$, $s_{14} = s_1 + s_4 = s_1$, $s_{23} = s_2 + s_3 = s_2$, $s_{24} = s_2 + s_4 = s_2$, $s_{34} = 0$, and that equation (32) can be written in the form

$$y'' + \left(\frac{1 - \rho_1 - \sigma_1}{z - a_1} + \frac{1 - \rho_2 - \sigma_2}{z - a_2}\right) y'$$
$$+ \left(\frac{\rho_1 \sigma_1}{(z - a_1)^2} + \frac{\rho_2 \sigma_2}{(z - a_2)^2} + \frac{\rho_{12}\sigma_{12} - \rho_1\sigma_1 - \rho_2\sigma_2}{(z - a_1)(z - a_2)}\right) y = 0. \tag{44}$$

Equation (44) is the Riemann differential equation (see [26]). Its solution can be presented via the Gauss hypergeometric function (see [26]), ${}_2F_1(a, b; c; z) = \sum\limits_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} z^n$, ($(q)_n$ is the Pochhammer symbol, $(q)_0 = 1$, $(q)_n = q(q-1)\cdots(q-n+1)$) with parameters $a = \rho_1 + \rho_2 - \rho_{12}, b = \rho_1 + \rho_2 - \sigma_{12}, c = 1 + \rho_1 - \sigma_1$.

From (40) it follows that we can take $\delta_1 = c$, $\varepsilon_1 = \frac{\mu_{21}^{(1)} \lambda_{11}^{(1)}}{\mu_{11}^{(1)} \lambda_{21}^{(1)}} c$, $\delta_2 = \frac{\lambda_{12}^{(1)}}{\mu_{12}^{(1)}} c$, $\varepsilon_2 = \frac{\lambda_{11}^{(1)}}{\mu_{11}^{(1)}} c$, where c is an arbitrary constant. Analogously, we can define connections between γ_k, δ_k, ε_k in those cases when at least one of the matrices $V_k, k = 1, 2$, is transformed to the triangular Jordan canonical form.

6. PARTIAL INDICES AND SOLUTION TO BOUNDARY VALUE PROBLEM

The Riemann–Hilbert problem can be formulated as the Riemann boundary value problem for analytic functions. To see this we draw a simple closed loop Γ through our singular points. Then bypassing the point a_k the following transformation yields $Y^+ \rightarrowtail V_1 \cdot V_2 \cdot \ldots \cdot V_k \cdot Y^+ = Y^-$. Hence we arrive at the boundary condition

$$Y^+(t) = A(t) Y^-(t), \quad t \in \Gamma \setminus \{a_1, a_2, \ldots, a_5\}, \tag{45}$$

where $A(t) = A_k = (V_1 V_2 \cdots V_k)^{-1}, t \in (a_k, a_{k+1}), A_n = E$ (note that we are looking for the solution to (45) unbounded near each singular point as stated in condition (iii) of the Riemann problem).

We must now recall some definitions from the theory of boundary value problems corresponding to the considered situation (see [24, § 126]). The matrix-function $X(z)$, analytic outside the contour Γ, is called the *canonical matrix for boundary value problem* (45) if it satisfies the following conditions:

1. $X^+(t) = A(t) X^-(t), t \in \Gamma \setminus \{a_1, a_2, \ldots, a_5\}$;
2. $\det X(z) \neq 0, \forall z \neq a_1, a_2, \ldots, a_5$;
3. the columns of the matrix $X(z)$ belong to the chosen class of solutions, namely, are supposed to be integrable in a neighbourhood of $a_1, \ldots, a_4$ (i.e. satisfy power type asymptotics similar to that in (iii) of Introduction) and almost bounded in a neighbourhood of $a_5 = \infty$;

4. the order of the determinant of $X(z)$ at infinity is equal to the sum of the orders at infinity of its columns.

Thus the columns of the canonical matrix are formed of the special system of the linear independent solutions to the boundary value problem (45) , installed as columns. The order of the determinant $\det X(z)$ of the canonical matrix at infinity is called *the index (or general index) of the boundary value problem* (45), and the integer parts of the orders of its columns are called *the partial indices*.

Theorem 3. *Let $u_k, v_k, k = 1, \ldots, 5$, be linear independent solutions of equation (32) in a neighbourhood of the corresponding singular point a_k, represented by one of the forms (33), (34). Let $p(z) = \prod_{k=1}^{4}(z - a_k), q(z) = r(z - b_1)(z - b_2)$ and b_1, b_2 be the roots of the quadratic equation (29). Then the matrix (37) is the canonical matrix of boundary value problem (45).*

The orders of the columns of (37) at infinity are equal

$$p_1 = \min\{\operatorname{Re}\rho, \operatorname{Re}\sigma\} = \operatorname{Re}\rho = \operatorname{Re}\rho_5 + \left[\frac{1-\Delta}{2}\right],$$

$$p_2 = \min\{\operatorname{Re}\rho, \operatorname{Re}\sigma - 1\} = \operatorname{Re}\sigma - 1 = \operatorname{Re}\rho_5 + \left[\frac{-\Delta}{2}\right],$$

where $\Delta = \sum_{k=1}^{5} \rho_k + \sigma_k$.

The partial indices of boundary value problem (45) are $\chi_1 = [p_1] = \left[\frac{1-\Delta}{2}\right], \chi_2 = [p_2] = \left[\frac{-\Delta}{2}\right]$, $0 \le \chi_1 \le 5, 0 \le \chi_2 \le 4, |\chi_2 - \chi_1| \le 1$.

The general index of boundary value problem (45) is equal $\chi = \chi_1 + \chi_2 = -\Delta, 0 \le \chi \le 9$.

Problem (45) has $l = \chi + 2$ linear independent solutions in the class of analytic vector-functions integrable in a neighbourhood of $a_1, \ldots, a_4$ (i.e. satisfy power type asymptotics similar to that in (iii) of Introduction) and almost bounded at a neighbourhood of $a_5 = \infty$, which can be found via the formula

$$Y(z) = X(z)\begin{pmatrix} P_{\chi_1}(z) \\ P_{\chi_2}(z) \end{pmatrix},$$

where $P_{\chi_1}(z)$, $P_{\chi_2}(z)$ are polynomials of orders χ_1, χ_2 respectively.

Corollary 3. *With the solution of the boundary value problem we can construct the solution of the factorization problem of the piecewise constant matrix $A(t), t \in \Gamma$. The partial indices of this factorization are χ_1, χ_2. It follows from Theorem 3 that the partial indices are stable.*

7. CONCLUSION

In this paper, we have proposed a method of construction of the solution of the Riemann–Hilbert problem in the case of several singular points. To avoid additional calculations we have restricted ourselves to the case of five singular points, but the method is universal and can be applied for larger numbers of points. During the course of the construction we have also determined the canonical matrix of the homogeneous boundary value problem with piecewise constant matrix coefficient. Thus the solvability conditions and solutions of the boundary value problem can be written in the standard way (see, e.g., [24]). Finally, the canonical matrix allows us to construct a solution for the factorization problem of the piecewise constant matrix (cf. [23, 8]).

ACKNOWLEDGMENTS

The work by SR is supported by the Simons Foundation Fellowship by Isaac Newton Mathematical Institute for Mathematical Sciences and by Belarusian Republician Foundation for Fundamental Research through grant F20R-083.

REFERENCES

1. T. Anselmo, R. Nelson, B. Carneiro da Cunha, and D. G. Crowdy, "Accessory parameters in conformal mapping: Exploiting the isomonodromic tau function for Painlevé VI," Proc. R. Soc. London, Ser. A **474**, 20180080 (2018).
2. V. I. Arnol'd and Yu. S. Il'yashenko, *Ordinary Differential Equations,* Vol. 1 of *Encyclopaedia Math. Sci.* (Springer, Berlin, 1988).
3. Y. Bibilo and G. Filipuk, "Constructive solutions to the Riemann-Hilbert problem and middle convolution," J. Dyn. Control Syst. **23**, 55–70 (2017).
4. A. A. Bolibruch, "The Riemann–Hilbert problem," Russ. Math. Surv. **45** (2), 1–47 (1990).
5. A. A. Bolibruch, *Inverse Monodromy Problems in the Analytic Theory of Differential Equations* (MTsNMO, Moscow, 2009) [in Russian].
6. W. Dekkers, "The matrix of a connection having regular singularities on a vector bundle of rank 2 on $\mathcal{P}^1(\mathbb{C})$," in *Équations différentielles et systèmes de Pfaff dans le champ complexe,* Lect. Notes Math. **712**, 33–43 (1979).
7. N. P. Erugin, *Riemann Problem* (Nauka Tekh., Minsk, 1982) [in Russian].
8. T. Ehrhardt and I. M. Spitkovsky, "Factorization of piece-wise constant matrix-functions and systems of differential equations," Algebra Anal. **13** (6), 56–123 (2001).
9. F. R. Gantmacher, *The Theory of Matrices* (AMS, Providence, RI, 2000).
10. D. Hilbert, *Grundzüge der Integralgleichungen. Drittes Abschnitt* (Leipzig, Berlin, 1912) [in German].
11. E. L. Ince and I. N. Sneddon, *Solution of Ordinary Differential Equations* (Longman, London, 1987).
12. L. A. Khvoshchinskaya, "Representation of the logarithm of the product of nonsingular matrices of the 2nd order," in *Proceedings of the 16th International Scientific Conference devoted to Acad. M. Krawtchuk, May 14–15, 2015* (KPI, Kiev, 2015), Vol. 2, pp. 192–195.
13. L. A. Khvoshchinskaya, "To the Riemann problem in the case of arbitrary number of points," in *Proceedings of the International Conference on Boundary Value Problems, Special Functions and Fractional Calculus* (BSU, Minsk, 1996), pp. 377–382.
14. L. A. Khvoshchinskaya, "Determination of access parameters at the solution of certain problems," Proc. Inst. Mat. NAN Belarus **9**, 145–150 (2001).
15. L. A. Khvoshchinskaya and T. N. Zhorovina, "Construction of the differential equation of a hydromechanical problem," in *Mathematical Methods in Engineering and Technologies, Proceedings of the International Scientific Conference MMTT-30* (Politekh. Univ., St. Petersburg, 2017), Vol. 3, pp. 14–18.
16. L. A. Khvoshchinskaya, "Solution of the Carleman type integral equation on a pair of intervals," in *Mathematical Methods in Engineering and Technologies, Proceedings of the International Scientific Conference MMTT-30* (Politekh. Univ., St. Petersburg, 2017), Vol. 3, pp. 9–13.
17. L. A. Khvoshchinskaya, "Construction of differential matrices of the Riemann problem for four and five singular points," in *Proceedings of the 6th Bogdanov Readings on Ordinary Differential Equations, December 7–10, 2015* (BSU, Minsk, 2015), pp. 46–48.
18. L. A. Khvoshchinskaya and T. N. Zhorovina, "Solution of the Carleman type integral equation on a pair of nonintersecting intervals," in *Mathematical Methods in Engineering and Technologies, Proceedings of the International Scientific Conference* (Politekh. Univ., St. Petersburg, 2018), Vol. 6, pp. 10–13.
19. L. A. Khvoshchinskaya and T. N. Zhorovina, "About one method of solving some problems of elasticity theory," in *Mathematical Methods in Engineering and Technology, Proceedings of the International Scientific Conference* (Politekh. Univ., St. Petersburg, 2019), Vol. 6, pp. 32–37.
20. L. A. Khvoshchinskaya and S. V. Rogosin, "On a solution method for the Riemann problem with two pairs of unknown functions," in *Analytic Methods of Analysis and Differential Equations,* Ed. by S. Rogosin and M. Dubatovskaya (Cambridge Scientific, Cottenham, Cambridge, UK, 2020), pp. 79–111.
21. A. T. Kohn, "Un résultat de Plemelj," in *Mathematics and Physics, 1979/1982* (Paris, 1983), pp. 307–312 [in French].
22. B. L. Krylov, "The solution in a finite form of the Riemann problem for a Gauss system," Tr. Kazan. Aviats. Inst. **31**, 203–445 (1956).
23. G. S. Litvinchuk and I. M. Spitkovsky, *Factorization of Measurable Matrix Functions* (Birkhäuser, Basel, Boston, 1987).
24. N. I. Muskhelishvili, *Singular Integral Equations,* 3rd ed. (Nauka, Moscow, 1968) [in Russian].
25. N. I. Muskhelishvili and N. P. Vekua, "The Riemann boundary problem for several unknown functions and its application to systems of singular integral equations," Tr. Tbil. Mat. Inst. **12**, 1–46 (1943).
26. NIST Handbook of Mathematical Functions, Ed. by F. W. J. Olver (NIST, Cambridge Univ. Press, 2010).
27. J. Plemelj, "Riemannsche Funktionenscharen mit gegebener Monodromiegruppe," Monatsh. Math. Phys. **19**, 211–245 (1908).
28. J. Plemelj, *Problems in the Sense of Riemann and Klein,* Vol. 16 of *Tracts Pure and Applied Mathematics* (Wiley, New York, 1964) [in German].

29. B. Riemann, *Grundlagen für eine allgemeine Theorie der Funktionen einer verändlichen komplexen Grösse. Werke* (Leipzig, 1876) [in German].
30. S. Rogosin and G. Mishuris, "Constructive methods for factorization of matrix-functions," IMA J. Appl. Math. **81**, 365–391 (2016).
31. N. P. Vekua, *Systems of Singular Integral Equations and Some Boundary Value Problems,* 2nd ed. (Nauka, Moscow, 1967) [in Russian].

Appendix

The proof of Proposition 1. Incorporating similarity of matrices $S_k \sim W_k$ we rewrite relation (11) in the form

$$\begin{pmatrix} \rho_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} s_1 & c \\ d & \rho_1+\sigma_1-s_1 \end{pmatrix} + \begin{pmatrix} s_2 & -c \\ -d & \rho_2+\sigma_2-s_2 \end{pmatrix}, \tag{46}$$

which is equivalent to the following system of four algebraic equations

$$\begin{cases} s_1+s_2=\rho_3, \\ \rho_1+\sigma_1+\rho_2+\sigma_2-s_1-s_2=\sigma_3, \\ s_1(\rho_1+\sigma_1-s_1)-cd=\rho_1\cdot\sigma_1, \\ s_2(\rho_2+\sigma_2-s_2)-cd=\rho_2\cdot\sigma_2. \end{cases} \tag{47}$$

We note that assumption (10) yields linear dependence of the first and second equations in (47). Hence we can determine only three parameters of system (47). Simple algebra yields the solution of this system

$$s_1=\frac{\rho_1\sigma_1-(\rho_3-\rho_2)(\rho_3-\sigma_2)}{\sigma_3-\rho_3}, \quad s_2=\frac{\rho_2\sigma_2-(\rho_3-\rho_1)(\rho_3-\sigma_1)}{\sigma_3-\rho_3}.$$

The product cd can be determined either from the third or fourth equation.

$$cd=s_1(\rho_1+\sigma_1-s_1)-\rho_1\sigma_1=-(s_1-\rho_1)(s_1-\sigma_1).$$

The first factor $s_1-\rho_1$ is

$$s_1-\rho_1=\frac{\rho_1\sigma_1-(\rho_3-\rho_2)(\rho_3-\sigma_2)-\rho_1(\sigma_3-\rho_3)}{\sigma_3-\rho_3}.$$

Using (10) we can present this in the form

$$s_1-\rho_1=\frac{\rho_1\sigma_1-\rho_3(\rho_1+\sigma_1-\sigma_3)-\rho_2\sigma_2-\rho_1(\sigma_3-\rho_3)}{\sigma_3-\rho_3}=\frac{(\rho_3-\rho_1)(\sigma_3-\sigma_1)-\rho_2\sigma_2}{\sigma_3-\rho_3}.$$

Similarly

$$s_1-\sigma_1=\frac{(\rho_3-\sigma_1)(\sigma_3-\rho_1)-\rho_2\sigma_2}{\sigma_3-\rho_3}.$$

By taking

$$c=(s_1-\sigma_1)a, \quad d=-(s_1-\rho_1)\frac{1}{a},$$

with arbitrary constant a we arrive at the final representation for matrices S_1 and S_2. □

The proof of Proposition 2. Let $\rho_4 \neq \sigma_4$. Then the matrix S_4 reduces to the diagonal Jordan form $S_4=\begin{pmatrix} \rho_4 & 0 \\ 0 & \sigma_4 \end{pmatrix}$, which can be represented by a sum of three matrices

$$S_4=S_1+S_2+S_3=\sum_{k=1}^{3}\begin{pmatrix} s_k & c_k \\ d_k & s'_k \end{pmatrix}, \tag{48}$$

where

$$S_k \sim W_k; \quad s'_k = \rho_k + \sigma_k - s_k, \quad c_k d_k = -(s_k - \rho_k)(s_k - \sigma_k), \quad k = 1, 2, 3,$$
$$\sum_{k=1}^{3} s_k = \rho_4, \quad \sum_{k=1}^{3} s'_k = \sigma_4, \quad \sum_{k=1}^{3} c_k = 0, \quad \sum_{k=1}^{3} d_k = 0.$$

The aim now becomes finding an explicit representation for the matrices $S_k, k = 1, 2, 3$.

Let us write the product $V_1 \cdot V_2 \cdot V_3$ in the following two forms

$$V_4 = V_1 \cdot V_2 \cdot V_3 = V_1 (V_2 \cdot V_3) = V_1 \cdot V_{23}, \tag{49}$$

$$V_4 = V_1 \cdot V_2 \cdot V_3 = (V_1 \cdot V_2) \cdot V_3 = V_{12} \cdot V_3, \tag{50}$$

denote by α_{23}, β_{23} and α_{12}, β_{12} the eigenvalues of the matrices V_{23} and V_{12}, respectively, and define the parameters

$$\rho_{23} = \frac{1}{2\pi i} \ln \alpha_{23}, \quad \sigma_{23} = \frac{1}{2\pi i} \ln \beta_{23}, \quad \rho_{12} = \frac{1}{2\pi i} \ln \alpha_{12}, \quad \sigma_{12} = \frac{1}{2\pi i} \ln \beta_{12},$$

where the branches of the logarithmic functions are fixed according to the following conditions,

$$\rho_{23} + \sigma_{23} = \rho_2 + \rho_3 + \sigma_2 + \sigma_3, |\mathrm{Re}\,(\rho_{23} - \sigma_{23})| < 1,$$
$$\rho_{12} + \sigma_{12} = \rho_1 + \rho_2 + \sigma_1 + \sigma_2, |\mathrm{Re}\,(\rho_{12} - \sigma_{12})| < 1.$$

Using representations (12), (13) to (49) and (50) we arrive at two formulas for S_4

$$S_4 = S_1 + S_2 + S_3 = S_1 + S_{23} = \begin{pmatrix} s_1 & c_1 \\ d_1 & s'_1 \end{pmatrix} + \begin{pmatrix} \rho_4 - s_1 & -c_1 \\ -d_1 & \sigma_4 - s'_1 \end{pmatrix}, \tag{51}$$

$$S_4 = S_1 + S_2 + S_3 = S_{12} + S_3 = \begin{pmatrix} \rho_4 - s_3 & -c_3 \\ -d_3 & \sigma_4 - s'_3 \end{pmatrix} + \begin{pmatrix} s_3 & c_3 \\ d_3 & s'_3 \end{pmatrix}, \tag{52}$$

where

$$s_1 = \frac{[\rho_1\sigma_1 - (\rho_4 - \rho_{23})(\rho_4 - \sigma_{23})]}{\sigma_4 - \rho_4} = \frac{[\rho_4(\sigma_4 - \rho_1 - \sigma_1) + \rho_1\sigma_1 - \rho_{23}\sigma_{23})]}{\sigma_4 - \rho_4}, \tag{53}$$

$$s_3 = \frac{[\rho_3\sigma_3 - (\rho_4 - \rho_{12})(\rho_4 - \sigma_{12})]}{\sigma_4 - \rho_4} = \frac{[\rho_4(\sigma_4 - \rho_3 - \sigma_3) + \rho_3\sigma_3 - \rho_{12}\sigma_{12})]}{\sigma_4 - \rho_4}. \tag{54}$$

Here $S_{12} \sim \frac{1}{2\pi i} \ln(V_1 V_2)$, $S_{23} \sim \frac{1}{2\pi i} \ln(V_2 V_3)$.

It follows from (51), (52) that $S_2 = S_{12} - S_1 = S_{23} - S_3$. Then, in particular,

$$\begin{pmatrix} s_2 & c_2 \\ d_2 & s'_2 \end{pmatrix} = \begin{pmatrix} \rho_4 - s_3 - s_1 & -c_3 - c_1 \\ -d_3 - d_1 & \sigma_4 - s'_3 - s'_1 \end{pmatrix}.$$

Hence

$$s_2 = \frac{[-\rho_4(\rho_2 - \sigma_2) - \rho_1\sigma_1 - \rho_3\sigma_3 + \rho_{12}\sigma_{12} + \rho_{23}\sigma_{23}]}{\sigma_4 - \rho_4}, \tag{55}$$

$$s_2 s'_2 - (c_1 + c_3)(d_1 + d_3) = \rho_2\sigma_2. \tag{56}$$

To simplify relation (55) we apply the following identity.

Lemma 1.

$$\det(S_1 + S_2 + S_3) = \det(S_1 + S_2) + \det(S_1 + S_3) + \det(S_2 + S_3)$$
$$- \det(S_1) - \det(S_2) - \det(S_3). \tag{57}$$

Proof. It follows from the elementary algebra and known relation for the determinant of the sum of two 2×2 matrices
$$\det(A+B)=\det(A)+\det(B)+\mathrm{tr}(A)\mathrm{tr}(B)-\mathrm{tr}(AB).$$
□

Let us denote
$$\omega_{23}:=\det(S_2+S_3)=\rho_{23}\sigma_{23}=\det\left(\frac{1}{2\pi i}\ln V_{23}\right),$$
$$\omega_{12}:=\det(S_1+S_2)=\rho_{12}\sigma_{12}=\det\left(\frac{1}{2\pi i}\ln V_{12}\right). \tag{58}$$
Then, it follows from (57) that
$$\omega_{13}:=\det(S_1+S_3)=\sum_{k=1}^{4}\rho_k\sigma_k-\omega_{23}-\omega_{12}. \tag{59}$$
Hence relation (55) can be rewritten in the form
$$s_2=\frac{[\rho_4(\sigma_4-\rho_2-\sigma_2)+\rho_2\sigma_2-\omega_{13}]}{\sigma_4-\rho_4}. \tag{60}$$

We now transform relation (56). We denote $\gamma_k:=-(s_k-\rho_k)(s_k-\sigma_k)$. Then, since $c_kd_k=s_ks'_k-\rho_k\sigma_k=\gamma_k$, we see from (56):
$$\gamma_2-\gamma_1-c_3d_1-c_1d_3-\gamma_3=0.$$
Hence, either
$$\gamma_2-\gamma_1-\gamma_3-\gamma_1\frac{c_3}{c_1}-\gamma_3\frac{c_1}{c_3}=0\quad(c_1\neq 0,c_3\neq 0), \tag{61}$$
or
$$\gamma_2-\gamma_1-\gamma_3-\gamma_3\frac{d_1}{d_3}-\gamma_1\frac{d_3}{d_1}=0\quad(d_1\neq 0,d_3\neq 0). \tag{62}$$
With $\tau=\frac{c_3}{c_1}$ in (61) or $\tau=\frac{d_3}{d_1}$ in (62) we obtain the following equation with respect to τ:
$$\gamma_1\tau^2-(\gamma_1+\gamma_3-\gamma_2)\tau+\gamma_3=0. \tag{63}$$
If $\gamma_1\neq 0$, then the solution to (63) has the form
$$\tau=\frac{1}{2\gamma_1}\left(\gamma_1+\gamma_3-\gamma_2\pm\sqrt{(\gamma_1^2+\gamma_2^2+\gamma_3^2)-2(\gamma_1\gamma_2+\gamma_1\gamma_3+\gamma_2\gamma_3)}\right). \tag{64}$$
If $\gamma_1=0$, then $\tau=\frac{\gamma_3}{\gamma_2-\gamma_3}$.

With known τ, we determine all entries c_k,d_k of the matrices $S_k, k=1,2,3$.

If $\tau=\frac{c_3}{c_1}$, then $c_1=c$, $c_3=\tau c$, $c_2=-(c_1+c_3)=-(1+\tau)c$, $d_1=\frac{\gamma_1}{c}$, $d_2=-\frac{\gamma_2}{(1+\tau)c}$, $d_3=\frac{\gamma_3}{c}$, where c is an arbitrary constant, $c\neq 0$.

If $\tau=\frac{d_3}{d_1}$, then $d_1=d$, $d_3=\tau d$, $d_2=-(d_1+d_3)=-(1+\tau)d$, $c_1=\frac{\gamma_1}{d}$, $c_2=-\frac{\gamma_2}{(1+\tau)d}$, $d_3=\frac{\gamma_3}{d}$, where d is an arbitrary constant, $d\neq 0$.

Therefore, we have obtained two representations of the Jordan canonical form of the logarithm of the product of three nonsingular 2×2 matrices as sum of matrices:
$$\begin{pmatrix}\rho_4 & 0\\ 0 & \sigma_4\end{pmatrix}=\begin{pmatrix}s_1 & c\\ \frac{\gamma_1}{c} & \rho_1+\sigma_1-s_1\end{pmatrix}+\begin{pmatrix}s_2 & -(1+\tau)c\\ \frac{-\gamma_2}{(1+\tau)c} & \rho_2+\sigma_2-s_2\end{pmatrix}+\begin{pmatrix}s_3 & \tau c\\ \frac{\gamma_3}{\tau c} & \rho_3+\sigma_3-s_3\end{pmatrix}, \tag{65}$$
$$\begin{pmatrix}\rho_4 & 0\\ 0 & \sigma_4\end{pmatrix}=\begin{pmatrix}s_1 & \frac{\gamma_1}{d}\\ d & \rho_1+\sigma_1-s_1\end{pmatrix}+\begin{pmatrix}s_2 & \frac{-\gamma_2}{(1+\tau)d}\\ -(1+\tau)d & \rho_2+\sigma_2-s_2\end{pmatrix}+\begin{pmatrix}s_3 & \frac{\gamma_3}{\tau d}\\ \tau d & \rho_3+\sigma_3-s_3\end{pmatrix}, \tag{66}$$

where the parameters $s_k, k = 1, 2, 3$, are defined in (53), (60), (54), respectively, τ is determined from equation (63), and c and d are arbitrary constants. □

The proof of Proposition 3. In the considered class of solutions, by Lemma 7 we have

$$\det(S_1 + S_2 + (S_3 + S_4)) = \omega_{12} + \omega_{234} + \omega_{134} - \omega_1 - \omega_2 - \omega_{34} =: \omega,$$
$$\det((S_1 + S_2) + S_3 + S_4) = \omega_{123} + \omega_{34} + \omega_{134} - \omega_{12} - \omega_3 - \omega_4 =: \omega.$$

Therefore, we have two relations, either

$$\omega_{134} = \omega + \omega_1 + \omega_2 + \omega_{34} - \omega_{12} - \omega_{234}, \tag{67}$$

$$\omega_{234} = \omega + \omega_3 + \omega_4 + \omega_{12} - \omega_{34} - \omega_{123}. \tag{68}$$

Let us write the first representation of the matrix S, corresponding to (20)

$$S_5 = \begin{pmatrix} s_1 & c_1 \\ \frac{\gamma_1}{c_1} & \rho_1 + \sigma_1 - s_1 \end{pmatrix} + \begin{pmatrix} s_2 & -(1+\tau_1)c_1 \\ -\frac{\gamma_2}{(1+\tau_1)c_1} & \rho_2 + \sigma_2 - s_2 \end{pmatrix} + \begin{pmatrix} s_{34} & \tau_1 c_1 \\ \frac{\gamma_{34}}{\tau_1 c_1} & \rho_{34} + \sigma_{34} - s_{34} \end{pmatrix}$$
$$= S_1 + S_2 + S_{34}, \tag{69}$$

where

$$s_1 = \frac{[\rho(\sigma + \rho_1 + \sigma_1 - 1) + \omega_1 - \omega_{234}]}{1 + \rho - \sigma},$$
$$s_2 = \frac{[\rho(\sigma + \rho_2 + \sigma_2 - 1) + \omega_2 - \omega_{134}]}{1 + \rho - \sigma},$$
$$s_{34} = \frac{[\rho(\sigma + \rho_{34} + \sigma_{34} - 1) + \omega_{34} - \omega_{12}]}{1 + \rho - \sigma},$$
$$\gamma_k = -(s_k - \rho_k)(s_k - \sigma_k), \quad k = 1, 2; \quad \gamma_{34} = -(s_{34} - \rho_{34})(s_{34} - \sigma_{34}),$$

the number τ_1 is a solution of equation (70)

$$\gamma_1 \tau_1^2 + (\gamma_1 + \gamma_{34} - \gamma_2)\tau_1 + \gamma_{34} = 0, \tag{70}$$

and c_1 is an arbitrary constant.

Let us write the second representation of the matrix S, corresponding to (20)

$$S_5 = \begin{pmatrix} s_{12} & c_2 \\ \frac{\gamma_{12}}{c_2} & \rho_{12} + \sigma_{12} - s_{12} \end{pmatrix} + \begin{pmatrix} s_3 & -(1+\tau_2)c_2 \\ -\frac{\gamma_3}{(1+\tau_2)c_2} & \rho_3 + \sigma_3 - s_3 \end{pmatrix} + \begin{pmatrix} s_4 & \tau_2 c_2 \\ \frac{\gamma_4}{\tau_2 c_2} & \rho_4 + \sigma_4 - s_4 \end{pmatrix}$$
$$= S_{12} + S_3 + S_4, \tag{71}$$

where

$$s_{12} = \frac{[\rho(\sigma + \rho_{12} + \sigma_{12} - 1) + \omega_{12} - \omega_{34}]}{1 + \rho - \sigma},$$
$$s_3 = \frac{[\rho(\sigma + \rho_3 + \sigma_3 - 1) + \omega_3 - \omega_{124}]}{1 + \rho - \sigma},$$
$$s_4 = \frac{[\rho(\sigma + \rho_4 + \sigma_4 - 1) + \omega_4 - \omega_{123}]}{1 + \rho - \sigma},$$
$$\gamma_{12} = -(s_{12} - \rho_{12})(s_{12} - \sigma_{12}); \quad \gamma_k = -(s_k - \rho_k)(s_k - \sigma_k), \quad k = 3, 4,$$

and τ_2 is a solution to equation (72)

$$\gamma_{12} \tau_2^2 + (\gamma_{12} + \gamma_4 - \gamma_3)\tau_2 + \gamma_4 = 0, \tag{72}$$

while c_2 is an arbitrary constant.

Comparing (69) and (71) we conclude that $S_1 + S_2 = S_{12}$ and $S_3 + S_4 = S_{34}$, i.e.

$$\begin{pmatrix} s_1+s_2 & -\tau_1 c_1 \\ \frac{1}{c_1}\left(\gamma_1+\frac{\gamma_2}{1+\tau_1}\right) & \rho_1+\sigma_1+\rho_2+\sigma_2-s_1-s_2 \end{pmatrix} = \begin{pmatrix} s_{12} & c_2 \\ \frac{\gamma_{12}}{c_2} & \rho_{12}+\sigma_{12}-s_{12} \end{pmatrix}, \tag{73}$$

$$\begin{pmatrix} s_3+s_4 & -c_2 \\ \frac{1}{c_2}\left(\frac{\gamma_4}{\tau_2}+\frac{\gamma_3}{1+\tau_2}\right) & \rho_3+\sigma_3+\rho_4+\sigma_4-s_3-s_4 \end{pmatrix} = \begin{pmatrix} s_{34} & \tau_1 c_1 \\ \frac{\gamma_{34}}{\tau_1 c_1} & \rho_{34}+\sigma_{34}-s_{34} \end{pmatrix}. \tag{74}$$

Now we will show that for $c_2 = -\tau_1 c_1$ relations (73) and (74) are satisfied identically. By Proposition 2,

$$s_1+s_2 = \frac{[\rho(\sigma-1)+\rho(\sigma+\rho_{12}+\sigma_{12}-1)+\omega_1+\omega_2-\omega_{134}-\omega_{234}]}{1+\rho-\sigma}$$
$$= \frac{[\rho(\sigma+\rho_{12}+\sigma_{12}-1)+\omega_{12}-\omega_{34}]}{1+\rho-\sigma} = s_{12}. \tag{75}$$

Analogously, it can be shown that $s_3 + s_4 = s_{34}$. Now we check that the following relations hold,

$$\frac{1}{c_1}\left(\gamma_1-\frac{\gamma_2}{1+\tau_1}\right) = \frac{\gamma_{12}}{c_2} \quad \text{and} \quad \frac{1}{c_2}\left(\frac{\gamma_4}{\tau_2}-\frac{\gamma_3}{1+\tau_2}\right) = \frac{\gamma_{34}}{\tau_1 c_1},$$

which, under the condition $c_2 = -\tau_1 c_1$, take the forms

$$\gamma_1-\frac{\gamma_2}{1+\tau_1} = \frac{\gamma_{12}}{-\tau_1} \quad \text{and} \quad -\frac{\gamma_4}{\tau_2}+\frac{\gamma_3}{1+\tau_2} = \gamma_{34}. \tag{76}$$

From (69) and (71) it follows that the following relations are valid

$$\gamma_1-\frac{\gamma_2}{1+\tau_1}+\frac{\gamma_{34}}{\tau_1} = 0 \quad \text{and} \quad \frac{\gamma_4}{\tau_2}-\frac{\gamma_3}{1+\tau_2}+\gamma_{12} = 0.$$

Direct calculations, however, show $\gamma_{12} = \gamma_{34}$. Hence equality (76) is satisfied identically.

Therefore, we get the following representation of matrix S_5:

$$S_5 = \begin{pmatrix} s_1 & c \\ \frac{\gamma_1}{c} & \rho_1+\sigma_1-s_1 \end{pmatrix} + \begin{pmatrix} s_2 & -(1+\tau_1)c \\ -\frac{\gamma_2}{(1+\tau_1)c} & \rho_3+\sigma_3-s_3 \end{pmatrix}$$
$$+ \begin{pmatrix} s_3 & \tau_1(1+\tau_2)c \\ \frac{\gamma_3}{\tau_1(1+\tau_2)c} & \rho_3+\sigma_3-s_3 \end{pmatrix} + \begin{pmatrix} s_4 & -\tau_1\tau_2 c \\ -\frac{\gamma_4}{\tau_1\tau_2 c} & \rho_4+\sigma_4-s_4 \end{pmatrix}$$
$$= S_1+S_2+S_3+S_4, \tag{77}$$

where τ_1, τ_2 are solutions to (70), (72), respectively, and c is an arbitrary constant. □