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RAY EXPANSION IN INHOMOGENEOUS AND STOCHASTIC MEDIA

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1. Rays' propagation in inhomogeneous media. Simulation of wave propagation processes in inhomogeneous media is based on kinematic [1-4] and dynamic principles [5-10] for the process of ray propagation and wave surfaces (fronts) in different media.

Huygens' principle of constructing wave fronts in according to algorithm of a contact transformation is easily implemented if the perturbation region is non-concave Fig. 1.1a

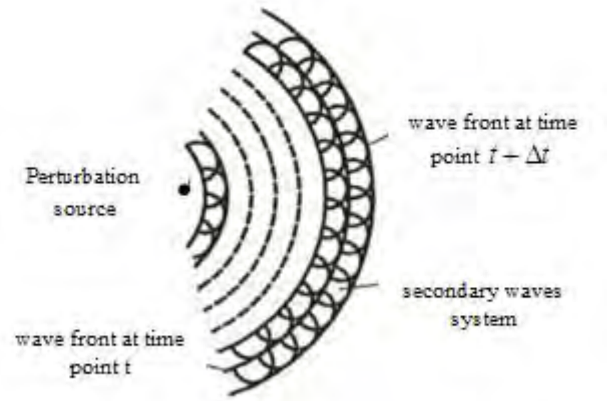

Fig. 1.1a. Huygens' model for propagation of wave fronts

If the emitting area has a concavity, the construction of the wave surface is shown in Fig. 1.1b

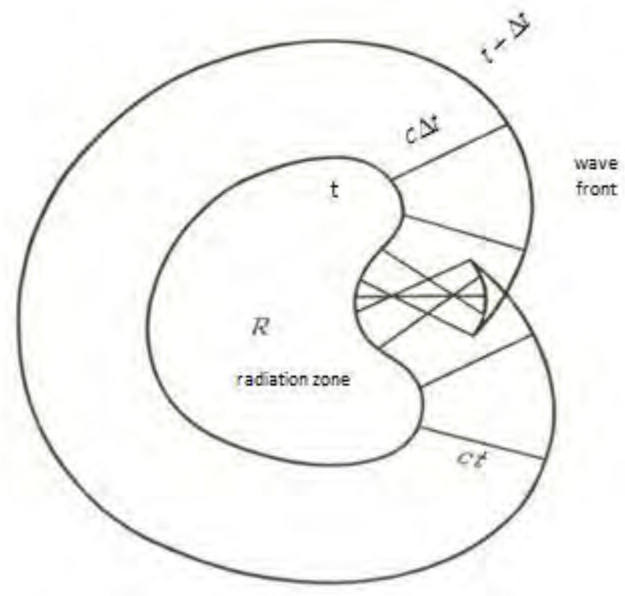

Fig. 1.1.b

Presentation of the wave front in the form of the surface is a mathematical idealization, because in reality, a wave is a volume configuration change of the medium points during the passage of a perturbation from some initial configuration to the final. In a homogeneous isotropic medium all front points have the same speed directed along the normal $\bar{n}$, then during the time Δt surface points are shifted by the same distance s along the normal with the speed $\bar{V}$. The wave surface at time point $t+\Delta t$, constructed according to Huygens' principle as the concavity of secondary waves coincides with the surface, passing through the points lying on one and the same distance along the normal from the wave surface at time point t. These lines, which are orthogonal to the original radiating surface (in particular, points) in a homogeneous isotropic medium, are the rays along which the radiation energy propagates.

According to **Newton's corpuscular theory**, the energy is transferred along the rays, the construction of which in homogeneous isotropic media is carried out with purely geometrical methods. Approaches of Huygens and Newton are known as the optic and mechanical analogs in analytical mechanics [1-2]. With the approach of Newton is associated the analogy of particle motion by inertia under the influence of the initial pulse and in the absence of any effects during the movement. With the approach of Huygens is associated the analogy of contact transformations in the Hamiltonian mechanics, representing a canonical transformation of generalized coordinates and momenta.

An approach based on the construction of rays is effective in solving problems of the wave kinematics by geometrical methods for homogeneous isotropic media, including the case of transmission and reflection of waves at the interface of two media, also through the lens, etc.

In the case of inhomogeneous, anisotropic, nonlinear media Huygens' approach is more difficult to implement and Newton's approach allows solving the problem of wave propagation more effectively, if we use the kinematic **principle of Farm**, according to which the perturbation of the medium state at the source $M_0(x_0, y_0, z_0)$ extends to any receiver point $M(x_1, x_2, x_3)$ for the minimum time $\tau(M_0, M)$, which is the Farm's functional [1-4]

$$\tau(M_0, M) = \int_{M_0}^{M} \frac{d\ell}{V(x_1, x_2, x_3)} \tag{1.1}$$

where $V(x_1, x_2, x_3)$ perturbation propagation speed including (x_1, x_2, x_3) that of inhomogeneous media, ℓ – the distance along the ray.

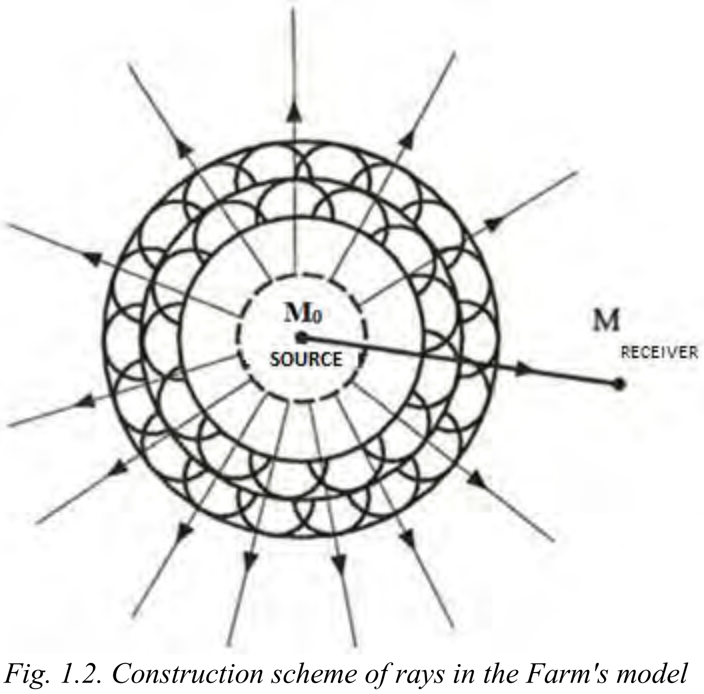

Fig. 1.2. Construction scheme of rays in the Farm's model

For equation (1.1) are formulated direct and inverse tasks.

In the direct task $V(x_1, x_2, x_3)$ is set and is possible to build surface-isochrones $\tau(M_0, M) = C$ (C – arbitrary constant), representing a family of wave fronts.

In the inverse task of the known $\tau(M_0, M)$ is necessary to determine $V(x_1, x_2, x_3)$, namely to identify the physical and mechanical characteristics of the media (media profile).

Ray tube in a inhomogeneous medium is a figure formed by adjacent rays Fig.1.3

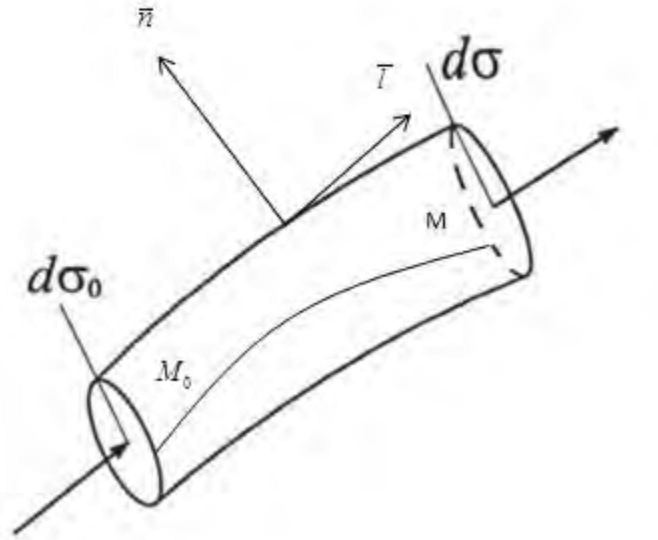

Fig. 1.3. Ray tube

We denote by A function that characterizes the change in the unit ray tube cross-sectional area of the value of $d\sigma_0$ at the initial front to the current cross section $d\sigma = A d\sigma_0, \overline{\ell}_0 = \nabla\tau / |\nabla\tau|$ – the unit vector directed along the ray Fig. 1.4.

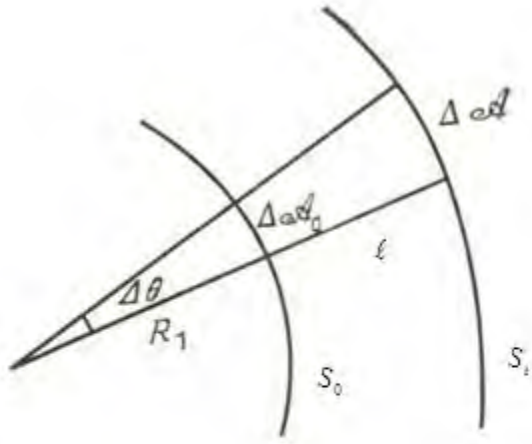

Fig. 1.4. The change σ in the ray

We calculate the integral by volume of ray tube Σ_t passing to the surface integral by the formula

$$\int_{\Sigma_t} div\left(\frac{\overline{\ell}}{A}\right) dv = \oint \frac{\overline{\ell}_0 * \overline{n}}{A} d\ell \tag{1.2}$$

Since on the side surface $\overline{\ell} \perp \overline{n}$, то $d\sigma A - d\sigma_0 A_0 = 0$, hence, we obtain

$$div\left[\left(v\nabla\tau\right)/A\right]=0,\ \nabla\tau=grad\,\tau \tag{1.3}$$

Considering the ratio of

$$v^2=\left|\nabla\tau\right|^{-2} \tag{1.4}$$

we obtain the equations for finding the position and shape of the wavefront. Equation (1.4) is called **the eikonal equation**, and the equation (1.3) determines the change in cross section of the tube.

In particular, for the case of 2D rays and consistent positions of the front form an orthogonal curvilinear grid α,β Fig. 1.5

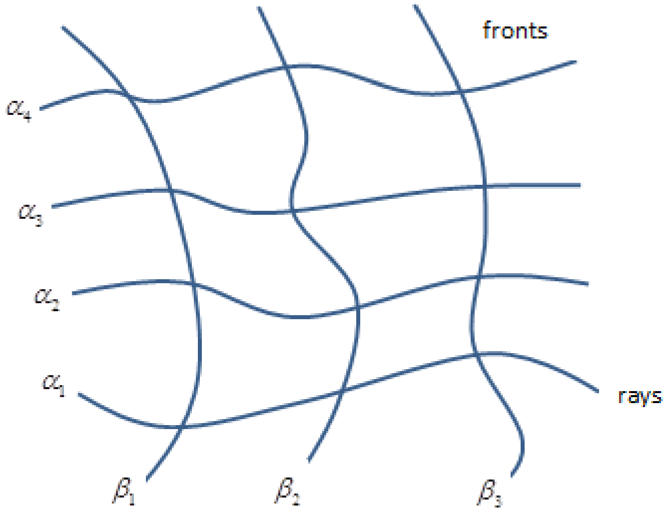

Fig. 1.5. The case of 2D rays

for which two kinematic equations are obtained

$$\frac{1}{V}\frac{\partial A}{\partial\alpha}=\frac{\partial\theta}{\partial\beta},\qquad \frac{1}{A}\frac{\partial V}{\partial\beta}=-\frac{\partial\theta}{\partial\alpha} \tag{1.5}$$

where $\theta(\alpha,\beta)$ – an angle which for linear homogeneous medium does not depend on α, because $V=V_0=const$. As A depends linearly on α, the rays will be straight, and the fronts will be circles in the plane considered.

The law of energy conservation in the integral form is

$$\frac{\partial}{\partial t}\int_{\Sigma_t}Edv=-\oint P_j n_j ds \tag{1.6}$$

Here P_j – **vector** components **of the energy density $\bar{P}$ of Poynting-Umov**, E – the total energy density.

In the differential form of (1.6) we obtain

$$\frac{\partial E}{\partial t}+div\bar{P}=0 \tag{1.7}$$

The energy flow is directed along the speed $\bar{V}$, and hence along the rays. For an arbitrary ray tube of (1.7) follows the conservation equation

$$\frac{\partial(EA)}{\partial t}+\frac{\partial\left(\bar{P}*\bar{A}\right)}{\partial e}=0 \tag{1.8}$$

where ℓ - the distance along the ray.

The eikonal equation (1.4) is a non-linear differential **equation in partial derivatives of the first order type Hamilton-Jacobi**, which is generally written as [11-14]

$$H\left(\frac{\partial\tau}{\partial q_1},\dots\frac{\partial\tau}{\partial q_n};q_1\dots q_n\right)=0 \tag{1.9}$$

where $\tau=\tau\left(q_1\dots q_n\right)$ – the desired function, q_i – the generalized coordinates $\left(i=\overline{1,n}\right)$.

Designating $\frac{\partial\tau}{\partial q_i}=p_i$ as generalized impulses, the equation (1.9) can be written as

$$H\left(p_i,q_i\right)=0 \tag{1.10}$$

Equation characteristics (1.9) satisfy the system of ordinary differential equations

$$\frac{dq_i}{\partial H/\partial p_i}=-\frac{dp_i}{\partial H/\partial q_i}=\frac{d\tau}{\sum_{i=1}^{n}p_j\left(\partial H/\partial p_i\right)} \tag{1.11}$$

which can be written as

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\ \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i},\ \frac{d\tau}{dt}=\sum_{j=1}^{n}p_j\frac{\partial H}{\partial p_j} \tag{1.12}$$

Here $2n$ component $q_i=q_i\left(t\right),p_i=p_i\left(t\right),$ which are solutions of the system (1.11) are called the characteristics of the equation (9).

If $q_i=x_i$ – the Cartesian coordinates, and $H=H\left(\overline{p},\overline{r}\right),\overline{r}=\left(x_1,x_2,x_3\right)$, the Hamiltonian characteristic equations (1.11) - (1.13) can be written in the vector form

$$\frac{d\overline{r}}{dt}=\frac{\partial H}{\partial\overline{p}},\ \frac{d\overline{p}}{dt}=-\frac{\partial H}{\partial\overline{r}} \tag{1.13}$$

$$\frac{d\tau}{dt}=\overline{p}\frac{\partial H}{\partial\overline{p}},\ \overline{p}=grad\,\tau=\nabla\tau \tag{1.14}$$

Differentiation respect to the vector, means differentiation with respect to the appropriate coordinate, for example $\frac{dx_i}{dt}=\frac{\partial H}{\partial\overline{p_i}}$. Thus, the ray is in the coordinate space of a projection $q_i=x_i=x_i\left(t\right)$ of the eikonal equation characteristics (1.4). In the phase space $\left\{p_i,q_i\right\}$ the characteristic $q_i=q_i\left(t\right),p_i=p_i\left(t\right)$ is called the ray equations.

If you found the solution of equations (1.13) in the form of $\overline{r}=r\left(t\right),\overline{p}=\overline{p}\left(t\right)$, then the solution of equation (1.14) can be written as

$$\tau=\tau_0+\int_{t_0}^{t}\overline{p}\frac{\partial H}{\partial\overline{p}}dt \tag{1.15}$$

There are different forms of recording the ray equations (1.13), depending on the kind of H, coordinate system selection. For example, we write H (eikonal equation) as

$$H=\frac{1}{2}\left[\overline{p}^2-n^2\left(\overline{r}\right)\right]=0,\qquad \overline{p}=grad\,\tau \tag{1.16}$$

where n – the refractive medium index.

Then the equations for the rays have the form of

$$\frac{d\overline{r}}{dt} = \overline{p} \tag{1.17}$$

$$\frac{d\overline{p}}{dt} = \frac{1}{2}\nabla n^2(r) = \frac{1}{2} grad\, n^2 \tag{1.18}$$

which shows that in an isotropic medium rays are orthogonal to the wave fronts.

Most convenient to use instead of the parameter t the parameter of the arc lengths of the curved ray in a inhomogeneous medium

$$dt = \frac{ds}{p} = \frac{ds}{n} \tag{1.19}$$

then the expression for the eikonal (1.16) has the form

$$\tau = \tau_0 + \int_{s_0}^{s} n(\overline{r})\, ds \tag{1.20}$$

If the eikonal equation (and) or H-equation of Hamilton-Jacobi (1.5) will be written in the form of

$$H(\overline{p},\overline{r}) = p - r(\overline{r}) = 0,\ p = \sqrt{\overline{p}^2} \tag{1.21}$$

then the rays equations (1.14) have the form

$$\frac{d\overline{r}}{ds} = \frac{\overline{p}}{n}, \qquad \frac{d\overline{p}}{dt} = \nabla n = grad\, n \tag{1.22}$$

and the expression for the eikonal (1.20).

To this same expression for the eikonal respond equations for the rays, written in the form

$$\frac{d\overline{r}}{ds} = \overline{\ell},\ \frac{d\overline{\ell}}{ds} = \frac{grad\, n}{n} - \overline{\ell}\left(\frac{grad\, n}{n}\overline{\ell}\right) = grad \perp \ln n \tag{1.23}$$

where $\overline{\ell} * grad\, n = dr/ds$ – derivative with respect to the ray, the operator $grad_{\perp} = grad - \overline{\ell}\left(\overline{\ell}\, grad\right)$ determines the gradient calculation in the direction perpendicular to the beam (along the wave front).

According to **the optical-mechanical analogy** the system of equations (1.14) can also be written in the form of Newtonian mechanics for potential forces (in the form of second-order equations)

$$\frac{d^2\overline{r}}{dt^2} = \frac{1}{2} grad\, n^2 \text{ or } \frac{d}{ds}\left(n\frac{d\overline{r}}{ds}\right) = grad\, n^2 \tag{1.24}$$

where the role of the forces potential is played by $n^2(\overline{r})$.

The geometry of the spatial curve (ray) is characterized by the **curvature *k* and torsion** $\varkappa$, which are calculated according to the formulas

$$k = \left|\left[\overline{\ell}\left(\frac{grad\, n}{n}\overline{\ell}\right)\right]\right| = \left|\left[\frac{grad\, n}{n}\overline{\ell}\right]\right| = \left|\frac{grad\, n}{n}\right| \tag{1.25}$$

where $0 < \theta < \frac{\pi}{2}$ – the angle between the ray (vector $\overline{\ell}\sin\theta$) and vector $grad\, n$.

The radius of the ray curvature $\rho = k^{-1}$. As in a homogeneous medium $n = const$, then $k = 0$, ie rays are straight lines.

The torsion is calculated according to the formula

$$\varkappa = \frac{1}{2}\left(\bar{n}\, rot\, \bar{n} + \bar{b}\, rot\, \bar{b}\right) \tag{1.26}$$

where $\bar{n}$ – the unit vector the main normal, $\bar{b} = \left[\bar{\ell} * n\right]$ the binormal unit vector in **the Frenet trihedron**, moving along the ray.

The expressions for $\bar{n}$ and $\bar{b}$ can be represented by the index of refraction $n(\bar{r})$ according to the formulas

$$\bar{n} = \frac{1}{k}\, grad_{\perp} \ln n,\ \ \bar{b} = \frac{1}{k}\left[\bar{\ell}\,\frac{grad\, n}{n}\right] \tag{1.27}$$

Then

$$\varkappa = \frac{1}{k}\left[\left(\bar{\ell}\, grad\right)\bar{b} * grad_{\perp} \ln n\right] = \frac{1}{k}\left[\left(\bar{\ell}\, grad\right) grad_{\perp} \ln n + \bar{b}\right]$$

For planar curves $\varkappa = 0$, for example in layered media.

Discussed equations correspond to coordinate method of setting a motion in mechanics. Natural way to set a motion in mechanics corresponds to the consideration of the kinematics of the rays in **the ray coordinates**, related to the initial position of the wave front [4,7,15].

On the surface of the radiating body curvilinear coordinates can be introduced ξ, η Fig. 1.6

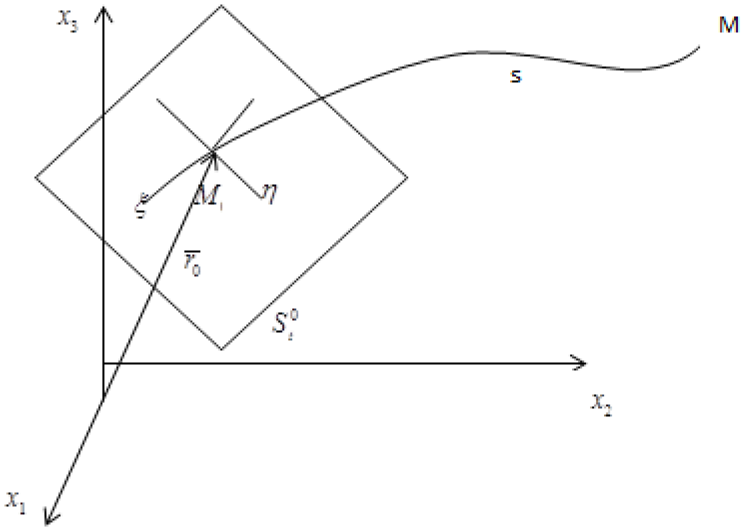

Fig. 1.6. The ray coordinates

so that

$$\bar{r}_0 = \bar{r}\left(t_0\right) = \bar{r}_0\left(\xi, \eta\right) \tag{1.28}$$

The coordinate lines ξ, η are orthogonal and they are chosen, as a rule, from geodesic lines or lines of the principal curvatures, coordinate line s – ray, tangent to which at ξ, η is orthogonal to the front S_t^0. Coordinate system ξ, η, s is called as **ray coordinate system**.

At $t = t_0$ you must set a condition for $\bar{p} = \bar{p}\left(t_0\right) = \bar{p}_0\left(\xi, \eta\right)$. The vector components $\bar{p}\left(\xi, \eta\right)$ satisfy the equations, which follow from the eikonal equation

$$\left(\overline{p}^0\right)^2 = n^2\left(\overline{r}_0\right),\ \overline{p}^0\frac{\partial\overline{r}^0}{\partial\xi},\ \overline{p}^0\frac{\partial\overline{r}_0}{\partial\xi} = \frac{\partial\overline{\tau}_0}{\partial\xi},\ \overline{p}^0\frac{\partial\overline{r}^0}{\partial\eta} = \frac{\partial\tau_0}{\partial\eta} \tag{1.29}$$

If the initial emitting surface is plane, then the coordinate system is a Cartesian system, and is connected with the surface $x_3^0 = 0$. Then assuming that $\xi = x_1^0, \eta = x_2^0$, we obtain

$$p_x^0 = \frac{\partial\tau_0}{\partial\xi},\ p_y^0 = \frac{\partial\tau_0}{\partial\eta},\ p_z^0 = \sqrt{n^2\left(\overline{r}_0\right) - \left(\frac{\partial\tau_0}{\partial\xi}\right)^2 - \left(\frac{\partial\tau_0}{\partial\eta}\right)^2} \tag{1.30}$$

The equation of the initial surface can be written as

$$x_i = x_i\left(u^\alpha, t\right),\ \alpha = 1,2; i = 1,2,3 \tag{1.31}$$

We introduce **the first and second quadratic form of the surface S_t^0**

$$g^{\alpha\beta} = x_i^\alpha * x_i^\beta,\ b^{\alpha\beta} = -\upsilon_\alpha^i * x_\beta^i = x_{\alpha\beta}^i \upsilon^i \tag{1.32}$$

Then the differential equations for the rays have the form [4,7,10]

$$\frac{\delta x^i}{\delta t} = c\upsilon^i,\ \ c = \upsilon^i\frac{\delta x^i}{\delta t} \tag{1.33}$$

$$\frac{\delta\upsilon^i}{\delta t} = -g^{\alpha\beta}x_\beta^i\upsilon^i\frac{\delta x_i^\alpha}{\delta t} = -g^{\alpha\beta}x_{i\beta}c_\alpha \tag{1.34}$$

To close the system (1.31), (1.32) it must be supplemented by equations for the mean and Gaussian curvatures of the surface.

In the particular case when the wave front propagates parallel to itself expressions for the mean and Gaussian curvatures have the form

$$\Omega = \frac{\Omega_0 - sK_0}{1 - 2s\Omega_0 + s^2K_0}\ \ K = \frac{K_0}{1 + s^2K_0 - 2sK_0} \tag{1.35}$$

The expression for the eikonal in the coordinates ξ, η, t is written in the form

$$\tau(\xi,\eta,t) = \tau_0(\xi,\eta) + \int_{t_0}^{t} n^2\left[\overline{r}(\xi,\eta,t)\right]dt \tag{1.36}$$

Coordinates ξ, η on the surface τ_0 identify a ray coming from the surface at the moment $t = t_0$. For different ξ, η we obtain a family of rays, therefore, at the initial time from the surface τ_0 comes the bundle of rays, that allows you to build a complex radiation pattern, on the basis of which is determined the wave field structure in the physical space.

In the phase space $\{\overline{p}, \overline{r}\}$ the phase portrait under certain conditions, can also be quite complicated.

In the spatial of generalized coordinates $\boldsymbol{q_i}$, which are not related to the wave surface and which are orthogonal curvilinear coordinates, eikonal can be written as

$$H\left(p_i, q_i\right) = \frac{1}{2}\sum_{i=1}^{3}\left\{\frac{p_j^2}{\mathrm{h}_j^2\left(q_i\right)} - n^2\left(q_i\right)\right\} \tag{1.37}$$

where $\mathbf{h}_j\left(\boldsymbol{q_i}\right)$ **– Lame coefficients** for curvilinear coordinates q_i.

The system of equations for the rays in this case has the form

$$\frac{dq_i}{dt}=\frac{1}{h_i^2}p_i,\ \frac{dp_i}{dt}=n\frac{\partial n}{\partial q_i}+\sum_{j=1}^{3}\frac{1}{h_j^2}\frac{\partial h_j}{\partial q_i}p_j^2 \tag{1.38}$$

The expression for the eikonal is written as

$$\tau=\tau_0+\int_{\tau_0}^{\tau}n^2\left[\overline{r}(t)\right]dt,\qquad p^2=n^2 \tag{1.39}$$

If you enter a pulse components in curvilinear coordinates according to the formulas

$$\hat{p}_i=\frac{P_i}{h_i}=\frac{1}{\mathrm{h}_i}\frac{\partial\tau}{\partial q_i},\ i=\overline{1,n} \tag{1.40}$$

then the equations (1.36) are written as

$$\frac{dq_i}{dt}=\frac{1}{\mathrm{h}_i}\hat{p}_i\ \frac{d\hat{p}_i}{dt}=\frac{1}{h_i}n\frac{\partial n}{\partial q_i}+\frac{1}{h_i}\sum_{j\neq i}^{3}\frac{\hat{p}_j}{h_j}\left(\hat{p}_j\frac{\partial h_j}{\partial q_i}-\hat{p}_i\frac{\partial h_i}{\partial qj}\right) \tag{1.41}$$

In many specific problems of ray propagation in inhomogeneous media is convenient to use **angular variables**, for example, in the case of the spherical symmetry. Assuming that $q_1=r,q_2=\theta,q_3=\varphi$ and considering that in this case $\mathrm{h}_1=1,\mathrm{h}_2=r,\mathrm{h}_3=r\sin\theta$, equations (1.39) can be written as

$$\frac{dr}{dt}=\hat{p}_r,\ \frac{d\theta}{dt}=\frac{1}{r}\hat{p}_\theta,\ \frac{d\varphi}{dt}=\frac{\hat{p}_\varphi}{r\sin\varphi},\ \frac{d\hat{p}_r}{dt}=n\frac{\partial n}{\partial r}+\frac{1}{r}\hat{p}_0^2+\frac{1}{r}\hat{p}_\varphi^2$$

$$\frac{d\hat{p}_\theta}{dt}=\frac{1}{r}\left(n\frac{\partial n}{\partial\theta}-\hat{p}_r\hat{p}_\theta+ctg\,j\theta*\hat{p}_\varphi^2\right) \tag{1.42}$$

$$\frac{d\hat{p}_\varphi}{dt}=\frac{1}{r\sin\theta}\left(n\frac{\partial n}{\partial\varphi}-\sin\theta\hat{p}_r\hat{p}_\theta-\cos\theta*\hat{p}_\theta\hat{p}_\varphi\right)$$

Here $\frac{\partial\tau}{\partial r}=\hat{p}_r,\ \frac{1}{r}\frac{\partial\tau}{\partial\theta}=\hat{p}_\theta,\ \frac{1}{r\sin\theta}\frac{\partial\tau}{\partial\theta}=\hat{p}_\varphi$.

The eikonal equation is written as

$$\left(\frac{\partial\tau}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\tau}{\partial\theta}\right)^2+\left(\frac{1}{r\sin\varphi}\frac{\partial\tau}{\partial\varphi}\right)^2=n^2 \tag{1.43}$$

or

$$\hat{p}_r^2+\hat{p}_\theta^2+\hat{p}_\varphi^2=n^2 \tag{1.44}$$

In the case of a plane task 2D- dimension, entering angle α by relations

$$\hat{p}_r=n\cos\alpha$$
$$\hat{p}_\theta=n\sin\alpha \tag{1.45}$$

we write the equations for the ray in the form of

$$\frac{d\theta}{dr}=\frac{1}{r}tg\,\alpha,\ \frac{d\alpha}{dr}=\frac{1}{nr}\left[\frac{\partial n}{\partial r}-\frac{\partial(rn)}{\partial r}\frac{d\theta}{dr}\right]\alpha=\pi/2 \tag{1.46}$$

Thus, depending on the geometry and mechanics of the particular task you can choose a suitable system of coordinates and shape of the ray equations. The most commonly used are phase coordinates $(\bar{r},\bar{p})$ and radial coordinates (ξ,η,s) so that the solution of the radial equation is represented as

$$\bar{r}=\bar{r}(\xi,\eta,t),\ \bar{p}=\bar{p}(\xi,\eta,t) \tag{1.47}$$

Here, the parameter t It related to the distance s along the ray (coordinate line), and ξ,η identify the ray on the initial surface S_t^0 at $t=t_0$. If the wave is emitted by a limited surface area, defined by the relation $\bar{r}^0=\bar{r}^0(\xi,\eta)$, then the rays form a family of rays emanating from this area.

The conversion from the Cartesian coordinate system to the ray system is defined by Jacobian $D=\dfrac{\partial(x_1,x_2,x_3)}{\partial(\xi,\eta,t)}$ and will be one to one, if $D\neq 0$.

A lot of wavefronts obtained in accordance with the principle of Huygens (contact transformations) form a family of equal phase surfaces, eikonal for each of them is written in the form

$$\tau(\xi,\eta,t)=\tau^0(\xi,\eta)+\int_{t_0}^{t}n^2\left[r(\xi,\eta,t)\right]dt=const=\tau_0\ \bar{r}=\bar{r}(\xi,\eta,t) \tag{1.48}$$

From the first equation (1.48) for each τ_0 can be found $t=t_\phi(\xi,\eta,t)$ and it is inserted into the second, then the family of wave fronts of equal phase (phase fronts) is determined by the ratio

$$\bar{r}=\bar{r}\left[\xi,\eta,t_\phi(\xi,\eta,\tau_0)\right]=r_\phi(\xi,\eta,\tau_0) \tag{1.49}$$

The family of rays emitted by the limited surface area S_t^0 forms a the bundle of rays. This means that the rays propagate not independent of each other, but they interfere. Due to the interference of secondary waves a significant contribution to the building of the fronts contribute only those rays, for which the phase difference does not differ by more than $\lambda/2$ (λ – wavelength).

Surfaces, where condition $D_{(t_K)|S_K}=0$ is violated are referred to as **caustic**

$$D_{(t_K)|S_K}=0 \qquad \text{at } \bar{r}\in\bar{r}_k(t) \tag{1.50}$$

The position of caustics is defined from the equation of the family of rays $\bar{r}=\bar{r}(\xi,\eta,t)$ and condition $D(t)=0$

$$\bar{r}=\bar{r}(\xi,\eta,t),\ D(\xi,\eta,t),=0 \tag{1.51}$$

Excluding t we obtain

$$\bar{r}=\bar{r}(\xi,\eta,t(\xi,\eta))=\bar{r}_k(\xi,\eta) \tag{1.52}$$

where $\bar{r}_k(\xi,\eta)$ determines the equation of the caustic in curvilinear coordinates of the initial surface. In solving problems for caustics it is convenient to introduce on the caustic surface own caustic coordinates (α,β,δ), where, α,β are curvilinear coordinates located on the caustic, and δ is measured along the line characterizing the removing from the caustic.

The value J Excluding t we obtain

$$J=\frac{D(t)}{D(t_0)}=\frac{\left(\left[\overline{r}_{,\xi}*\overline{r}_{,\eta}\right]*\overline{p}\right)}{\left(\left[\overline{r}_{,\xi}*\overline{r}_{,\eta}\right]*\overline{p}\right)_{t=t_0}}\ \overline{r}_{,\xi}=\frac{\partial\overline{r}}{\partial\xi},\ \overline{r}_{,\eta}=\frac{\partial\overline{r}}{\partial\eta},\ \overline{r}_{,t}=\frac{\partial\overline{r}}{\partial t}=\overline{p} \tag{1.53}$$

is called **the divergence of rays**.

From (1.51) follows that on the caustic $J=0$, ie cross-section of the ray tube decreases, energy increases, the rays touch caustics and change the direction. The classification of caustics is considered in the catastrophe theory [16]. On caustics and in their neighborhood classical spatial ray solutions are not applicable. There are methods for caustic rays, allowing to solve a number of tasks for caustics [2].

The wave fronts pass through **the interface between the physical and mechanical properties** of the medium, wherein the interface can have a complicated geometry. The study of the interaction of waves with obstacles (diffraction waves) with the help of radiation methods is considered in **the geometric theory of diffraction** [ ]. Using ray method in this case is based on the principle of locality, whereby in the neighborhood of each interface point the incident, reflected and refracted waves can be considered plane waves.

On Fig. 1.7 is shows a classic scheme of wave incidence on the interface between two media with different physical and mechanical properties. The plane of ray incidence contains vectors: of normal $\overline{N}$, incident $\overline{p}_{inc}$, refracted p_{refr} waves. At the interface the following conditions must be satisfied

$$\tau_{inc}=\tau_{refl}=\tau_{refr}\ \text{at}\ r\in Q \tag{1.54}$$

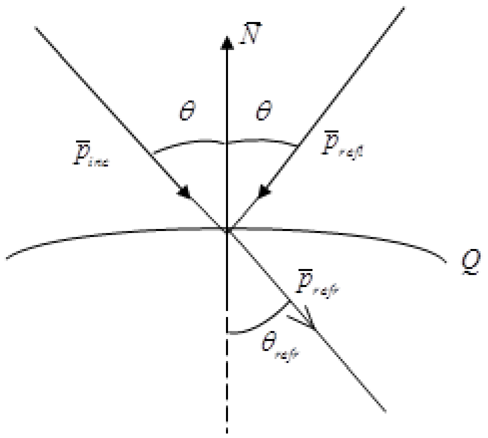

Fig. 1.7. A scheme of wave incidence on the interface between two media

In view of the conditions (1.53) tangential to the interface line Q components of the vectors have the form of

$$\overline{p}_{inc}=\nabla\tau_{inc}=grad\,\tau_{inc}.,\overline{p}_{refl}=\nabla\tau_{refl}=grad\,\tau_{refl}.,\qquad \overline{p}_{refr}=\nabla\tau_{refr}=grad\,\tau_{refr}$$
$$\left(\overline{p}_{inc}\right)_t=\left(\overline{p}_{refl}\right)_t=\left(\overline{p}_{refr}\right)_\tau,\overline{p}_\tau=\overline{p}-\overline{N}\left(p\overline{N}\right) \tag{1.55}$$

Here by the index τ tangential vectors are indicated for normal components of these vectors, representing at the same time normal derivatives of eikonals $\tau_{inc},\tau_{refl},\tau_{refr}$. We have the relations

$$\left(\overline{p}_{inc}\right)_N=-\sqrt{n_1^2-\left(\overline{p}_{inc}\right)_t^2},\ \left(\overline{p}_{refl}\right)_N=\sqrt{n_1^2-\left(p_{refl}\right)_t^2},\ \left(\overline{p}_{refr}\right)_N=\sqrt{n_2^2-\left(p_{inc}\right)_t^2},\ \text{at}\ \overline{r}\in Q \tag{1.56}$$

Designating with $\theta, \theta_{refl}, \theta_{refr}$ respectively, **the angles of incidence, reflection, refraction,** we write the equation in the form

$$n_1 \sin\theta = n_1 \sin\theta_{refl} = n_2 \sin\theta_{refr} \tag{1.57}$$

ie must be performed **mirroring laws**

$$\theta_{refl} = \theta \tag{1.58}$$

and the law of refraction (Snellius law)

$$\frac{\sin\theta}{\sin\theta_{refr}} = \frac{n_2}{n_1} \tag{1.59}$$

With these equations eikonal normal derivatives can be written as

$$\frac{\partial\tau_{refl}}{\partial N} = -\frac{\partial\tau_{inc}}{\partial N} = n_1\cos\theta, r\epsilon Q \quad \frac{\partial\tau_{refr}}{\partial N} = -n_1\cos\theta_{refr} = -\sqrt{n_2^2 - n_1^2\sin^2\theta} \tag{1.60}$$

Ratios (1.52) – (1.57) give all the necessary formulas for finding rays and eikonals of reflected and refracted waves.

The surface waves can exist at the interface of two media. The field of these waves decreases exponentially by leaving at the normal from the surfaces along which they propagate. At a smooth change of the geometrical and mechanical properties of the surface, in the scale of the wavelength, you can build **surface rays and fronts** [ ].

At a waves' diffraction on the bodies of different geometry diffraction rays propagate in the shadow zone behind obstacles, which are divided into two main types:

1. Edge rays, sources of which are the fins (edges), and the tips on the bodies Fig. 1.8

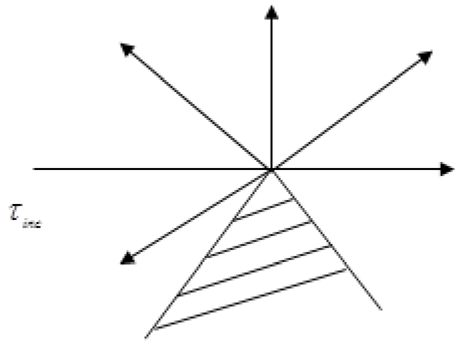

Fig. 1.8. Edge rays

2. Slipping rays (creeping rays) Fig.1.9

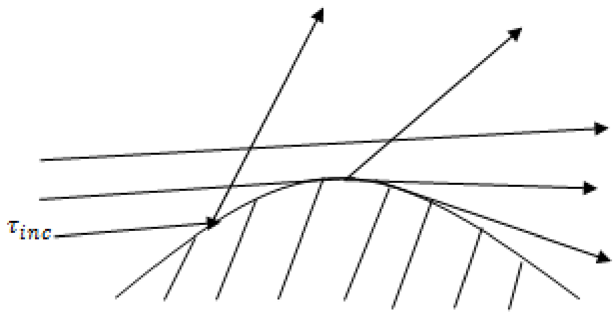

Fig. 1.9. Slipping rays

3. Diffraction rays of the lateral wave [2,8] in the presence of the refractive of the interface Fig.1.10

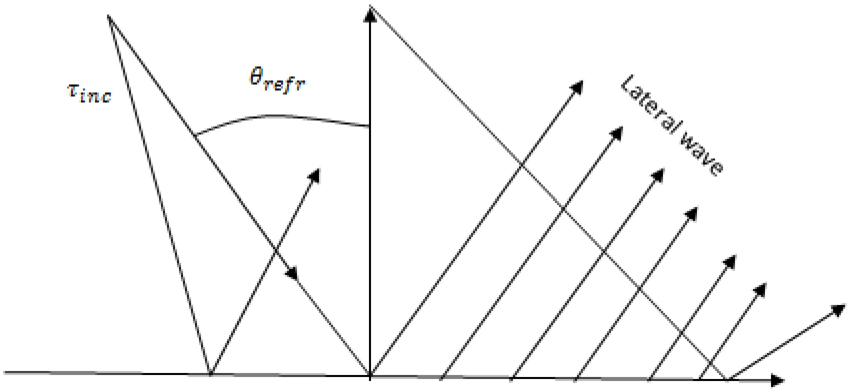

Fig. 1.10. Diffraction rays

4. Complex rays for waves with a complex eikonal, with which you can build rays in the caustic shadow zone, surface and leaky waves, etc. Fig. 1.11

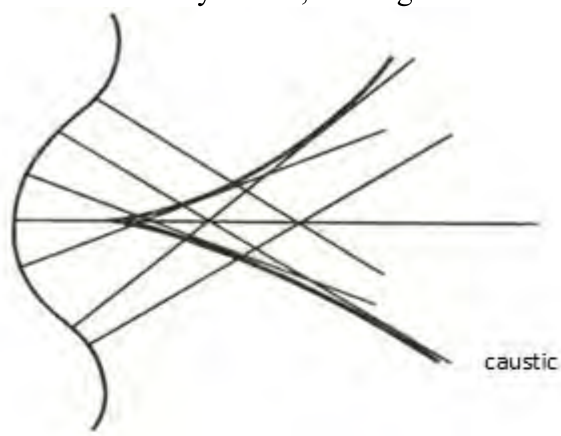

Fig. 1.11. The rays in the caustic zone

An extension of this kinematics of the spatial rays are **the space-time rays**, resulting in tasks of stationary waves' propagation, rays in anisotropic stationary media with temporal and spatial dispersion. The kinematics of the different types of space-time rays (complex, edge, etc.), which are related to the space-time geometrical theory of diffraction can be considered.

Due optic and mechanical analogy the considered spatial rays correspond in the analytical mechanics with scleronomic systems, and the theory of space-time rays is similar respectively to the rheonomic systems [2].

To be continued.